Calculus 1, Sections 1-5, Exam 2, Version C

Monday, November 17, 2008

1. (8 pts) Short answer. Put your answer in the blank. No explanation needed and **NO PARTIAL CREDIT!**

   (a) Evaluate $\int \sqrt[3]{x}\,dx$. $\boxed{} + C$

   Solution: $\frac{3}{4}x^{\frac{4}{3}} + C$.

   (b) Evaluate $G'(x)$ for $G(x) = \int_0^x \sin t\,dt$. $\boxed{}$

   Solution: $\sin x$.

   (c) Compute $\int (x^5 - x^2)\,dx$. $\boxed{} + C$

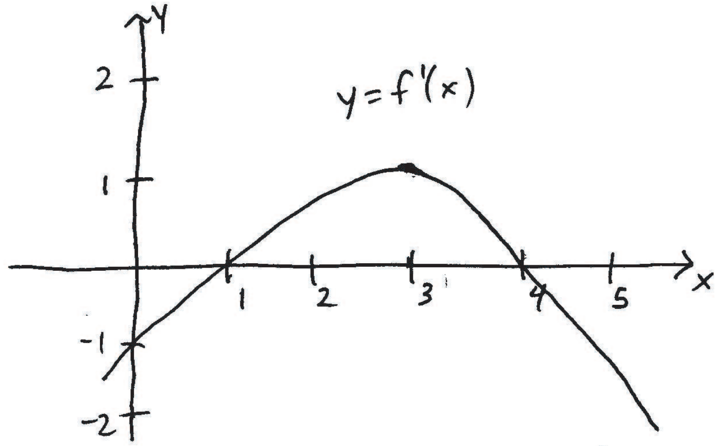
   Solution: $\frac{1}{6}x^6 - \frac{1}{3}x^3 + C$.

   (d) Compute the sum $\sum_{n=1}^{4}(n^2-1)$. Your answer should be in the form of *an integer*. $\boxed{}$

   Solution: $(1^2-1)+(2^2-1)+(3^2-1)+(4^2-1) = 0+3+8+15 = 26$.

2. (3 pts) Identify the critical points and find the maximum value and the minimum value for $f(x) = x^2 - 2x + 2$ on the interval $[0, 3]$. Show your work.

   Solution: Compute $f'(x) = 2x - 2$. $f'(x) = 0$ only when $x = 1$, which is in the interval $[0, 3]$. So we have three critical points: two endpoints $x = 0, 3$, and the stationary critical point $x = 1$. Compute $f(0) = 2$, $f(1) = 1$, $f(3) = 5$. So the minimum is at $x = 1$ with value 1, and the maximum is at $x = 3$ with value 5.



3. (8 pts) The figure above is a graph of *the derivative function* $y = f'(x)$. **BE SURE TO NOTE THE GRAPH ABOVE IS THE GRAPH OF $y = f'(x)$, NOT THE GRAPH OF $y = f(x)$.**

   (a) (3 pts) Find all local minimum and maximum points of $f(x)$ on the interval $[0, 5]$. (You should include the endpoints of the interval as possible local maxima and minima.)

   Solution: Local minimum points: $x = 1, 5$. Local maximum points $x = 0, 4$.

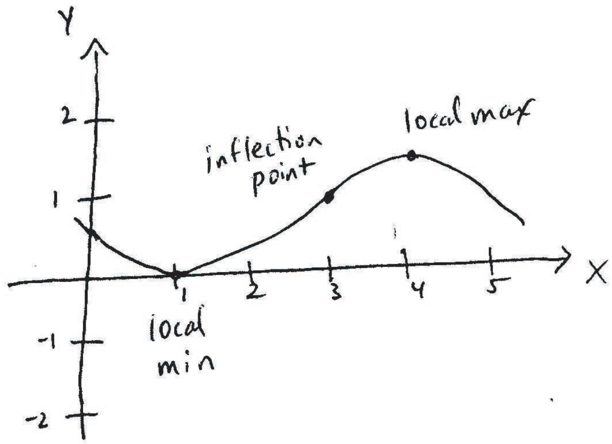
   First the endpoints: $f(x)$ is decreasing once it leaves $x = 0$, and so $x = 0$ is a local maximum. On the other hand, $f(x)$ is decreasing as it approaches $x = 5$, and so $x = 5$ is a local minimum.

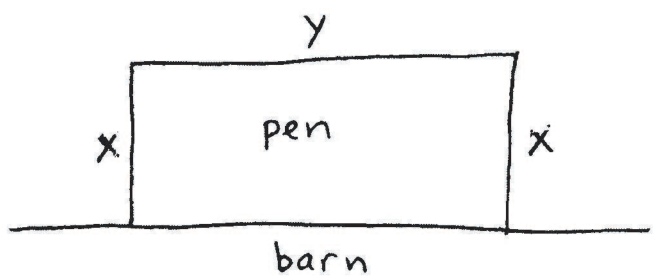
   For the stationary critical points, the graph shows $f'(x) = 0$ when $x = 1$ and when $x = 4$. Use the First Derivative Test to show $x = 1$ is a local minimum and $x = 4$ is a local maximum.

   (b) (2 pts) Find all points of inflection of $f(x)$ on the interval $[0, 5]$.

   Solution: $x = 3$ is the only inflection point. f goes from concave up to concave down there (since f' changes from increasing to decreasing there).

   (c) (3 pts) On the axes provided below, sketch the graph of $y = f(x)$, assuming that $f(1) = 0$. Be sure your graph reflects the information about the intervals on which $f(x)$ is increasing, decreasing, concave up, and concave down.





4. (4 pts) A farmer is building a rectangular pen against the side of a long barn, as in the picture above. If the area of the pen is to be 5000 ft^2, what is the minimum amount of fencing material (measured in ft) that the farmer must use? Show your work.

Solution: Let x be the length of the sides of the pen perpendicular to the barn, and let y be the length of the side of the pen parallel to the barn, as marked in the picture above. Then the total amount of fencing needed is $2x + y$. The area of the pen is fixed at $xy = 5000$ ft^2. So we can solve to find $y = \frac{5000}{x}$ and the amount of fencing needed is

$$f(x) = 2x + \frac{5000}{x}.$$

The natural domain of f is x in $(0, \infty)$. Compute $f'(x) = 2 - \frac{5000}{x^2}$. $f'(x) = 0$ when $2 = \frac{5000}{x^2}$, or $x^2 = 2500$, $x = 50$ (we only need to consider positive x). Thus $x = 50$ is the only critical point.

Compute $f''(x) = \frac{10000}{x^3}$ and so $f''(50) > 0$. The second derivative test shows $x = 50$ is a local minimum, and it must be the global minimum since it's the only critical point in the interval.

So the total amount of fencing material needed is

$$2x + y = 2x + \frac{5000}{x} = 2(50) + \frac{5000}{50} = 200\,\text{ft}.$$

5. (a) (3 pts) Find the general solution to the differential equation $\frac{dy}{dx} = x^2y^2$. Show your work.

Solution: Compute

$$\begin{aligned} \frac{dy}{dx} &= x^2y^2, \\ \frac{dy}{y^2} &= x^2\,dx, \\ \int \frac{dy}{y^2} &= \int x^2\,dx, \\ -\frac{1}{y} &= \frac{1}{3}x^3 + C, \\ y &= -\frac{1}{\frac{1}{3}x^3 + C} \\ &= -\frac{3}{x^3 + 3C}. \end{aligned}$$

(b) (3 pts) Find the particular solution to $\frac{dy}{dx} = x^2y^2$ which passes through the point $(x, y) = (1, 2)$. Show your work.

Solution: Plug in $(x, y) = (1, 2)$ to the general solution above to find

$$\begin{aligned} 2 &= -\frac{3}{1^3 + 3C}, \\ -\frac{3}{2} &= 1 + 3C, \\ C &= -\frac{5}{6}. \end{aligned}$$

So the particular solution is $y = -\frac{3}{x^3 + 3(-\frac{5}{6})} = -\frac{3}{x^3 - \frac{5}{2}}$.

6. (7 pts) Consider the function $h(\theta) = 3\tan\theta - 4\theta$ for θ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(a) (3 pts) Find all the critical points of $h(\theta)$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Show your work.

Solution: First note $h(\theta)$ is continuous on the given interval (the tangent function has vertical asymptotes at $\theta = \pm\frac{\pi}{2}$.) Now compute the derivative $h'(\theta) = 3\sec^2\theta - 4$. So $h'(\theta) = 0$ when

$$\begin{aligned} 0 &= 3\sec^2\theta - 4, \\ \sec^2\theta &= \frac{4}{3}, \\ \frac{1}{\cos^2\theta} &= \frac{4}{3}, \\ \cos^2\theta &= \frac{3}{4}, \\ \cos\theta &= \pm\frac{\sqrt{3}}{2}, \\ \theta &= \pm\frac{\pi}{6}. \end{aligned}$$

So the only critical points in the interval are $\theta = \frac{\pi}{6}$, $\theta = -\frac{\pi}{6}$.

(b) (2 pts) Classify each critical point from part (a) as a local minimum or a local maximum. Justify your answers.

Solution: Use the Second Derivative Test. Compute $h''(\theta) = 2\sec\theta(\sec\theta\tan\theta) = 2\sec^2\theta\tan\theta$. So $h''(\frac{\pi}{6}) = 2\sec^2\frac{\pi}{6}\tan\frac{\pi}{6} = 2(\frac{2}{\sqrt{3}})^2 \cdot \frac{1}{\sqrt{3}} > 0$, and so $\frac{\pi}{6}$ is a local minimum. On the other hand, $h''(-\frac{\pi}{6}) = 2\sec^2(-\frac{\pi}{6})\tan(-\frac{\pi}{6}) = 2(\frac{2}{\sqrt{3}})^2 \cdot (-\frac{1}{\sqrt{3}}) < 0$. So $-\frac{\pi}{6}$ is a local maximum.

(c) (2 pts) Does $h(\theta)$ have a global maximum point on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$? Why or why not?

Solution: $h(\theta)$ does not have a global maximum point on this interval since

$$\lim_{\theta\to\frac{\pi}{2}^-} h(\theta) = \lim_{\theta\to\frac{\pi}{2}^-} (3\tan\theta - 4\theta) = 3(\infty) - 4(\frac{\pi}{2}) = \infty.$$

So this infinite limit is larger than the value of h at the local max at $\theta = -\frac{\pi}{6}$.