1. (8 pts) Short answer. Put your answer in the blank. No explanation needed and NO PARTIAL CREDIT!

(a) Evaluate $\int 3^{\sqrt{x}} \, dx$.

Solution: $\frac{3}{4} x^{\frac{4}{3}} + C$. 

(b) Evaluate $G'(x)$ for $G(x) = \int_{0}^{x} \sin t \, dt$.

Solution: $\sin x$.

(c) Compute $\int (x^5 - x^2) \, dx$.

Solution: $\frac{1}{6} x^6 - \frac{1}{3} x^3 + C$.

(d) Compute the sum $\sum_{n=1}^{4} (n^2 - 1)$. Your answer should be in the form of an integer.

Solution: $(1^2 - 1) + (2^2 - 1) + (3^2 - 1) + (4^2 - 1) = 0 + 3 + 8 + 15 = 26$.

2. (3 pts) Identify the critical points and find the maximum value and the minimum value for $f(x) = x^2 - 2x + 2$ on the interval $[0, 3]$. Show your work.

Solution: Compute $f'(x) = 2x - 2$. $f'(x) = 0$ only when $x = 1$, which is in the interval $[0, 3]$. So we have three critical points: two endpoints $x = 0, 3$, and the stationary critical point $x = 1$. Compute $f(0) = 2$, $f(1) = 1$, $f(3) = 5$. So the minimum is at $x = 1$ with value 1, and the maximum is at $x = 3$ with value 5.
3. (8 pts) The figure above is a graph of the derivative function $y = f'(x)$. BE SURE TO NOTE THE GRAPH ABOVE IS THE GRAPH OF $y = f'(x)$, NOT THE GRAPH OF $y = f(x)$.

(a) (3 pts) Find all local minimum and maximum points of $f(x)$ on the interval $[0, 5]$. (You should include the endpoints of the interval as possible local maxima and minima.)

**Solution:** Local minimum points: $x = 1, 5$. Local maximum points $x = 0, 4$.

First the endpoints: $f(x)$ is decreasing once it leaves $x = 0$, and so $x = 0$ is a local maximum. On the other hand, $f(x)$ is decreasing as it approaches $x = 5$, and so $x = 5$ is a local minimum.

For the stationary critical points, the graph shows $f'(x) = 0$ when $x = 1$ and when $x = 4$. Use the First Derivative Test to show $x = 1$ is a local minimum and $x = 4$ is a local maximum.

(b) (2 pts) Find all points of inflection of $f(x)$ on the interval $[0, 5]$.

**Solution:** $x = 3$ is the only inflection point. $f$ goes from concave up to concave down there (since $f'$ changes from increasing to decreasing there).

(c) (3 pts) On the axes provided below, sketch the graph of $y = f(x)$, assuming that $f(1) = 0$. Be sure your graph reflects the information about the intervals on which $f(x)$ is increasing, decreasing, concave up, and concave down.
4. (4 pts) A farmer is building a rectangular pen against the side of a long barn, as in the picture above. If the area of the pen is to be 5000 ft$^2$, what is the minimum amount of fencing material (measured in ft) that the farmer must use? Show your work.

**Solution:** Let $x$ be the length of the sides of the pen perpendicular to the barn, and let $y$ be the length of the side of the pen parallel to the barn, as marked in the picture above. Then the total amount of fencing needed is $2x + y$. The area of the pen is fixed at $xy = 5000$ ft$^2$. So we can solve to find $y = \frac{5000}{x}$ and the amount of fencing needed is

$$f(x) = 2x + \frac{5000}{x}.$$ 

The natural domain of $f$ is $x$ in $(0, \infty)$. Compute $f'(x) = 2 - \frac{5000}{x^2}$. $f'(x) = 0$ when $2 = \frac{5000}{x^2}$, or $x^2 = 2500$, $x = 50$ (we only need to consider positive $x$). Thus $x = 50$ is the only critical point.
Compute $f''(x) = \frac{10000}{x^3}$ and so $f''(50) > 0$. The second derivative test shows $x = 50$ is a local minimum, and it must be the global minimum since it’s the only critical point in the interval.

So the total amount of fencing material needed is

$$2x + y = 2x + \frac{5000}{x} = 2(50) + \frac{5000}{50} = 200 \text{ ft}.$$ 

5. (a) (3 pts) Find the general solution to the differential equation $\frac{dy}{dx} = x^2 y^2$. Show your work.

**Solution:** Compute

$$\frac{dy}{dx} = x^2 y^2,$$

$$\frac{dy}{y^2} = x^2 \, dx,$$

$$\int \frac{dy}{y^2} = \int x^2 \, dx,$$

$$-\frac{1}{y} = \frac{1}{3} x^3 + C,$$

$$y = -\frac{1}{\frac{1}{3} x^3 + C}$$

$$= -\frac{3}{x^3 + 3C}.$$ 

(b) (3 pts) Find the particular solution to $\frac{dy}{dx} = x^2 y^2$ which passes through the point $(x, y) = (1, 2)$. Show your work.

**Solution:** Plug in $(x, y) = (1, 2)$ to the general solution above to find

$$2 = -\frac{3}{1^3 + 3C},$$

$$-\frac{3}{2} = 1 + 3C,$$

$$C = -\frac{5}{6},$$

So the particular solution is $y = -\frac{3}{x^3 + 3\left(-\frac{5}{6}\right)} = -\frac{3}{x^3 - \frac{5}{2}}$. 


6. (7 pts) Consider the function \( h(\theta) = 3 \tan \theta - 4\theta \) for \( \theta \) in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\).

(a) (3 pts) Find all the critical points of \( h(\theta) \) in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\).

Solution: First note \( h(\theta) \) is continuous on the given interval (the tangent function has vertical asymptotes at \( \theta = \pm \frac{\pi}{2} \)). Now compute the derivative \( h'(\theta) = 3 \sec^2 \theta - 4 \). So \( h'(\theta) = 0 \) when

\[
0 = 3 \sec^2 \theta - 4,
\]

\[
\frac{\sec^2 \theta}{4} = \frac{4}{3},
\]

\[
\frac{1}{\cos^2 \theta} = \frac{4}{3},
\]

\[
\frac{\cos^2 \theta}{3} = \frac{3}{4},
\]

\[
\cos \theta = \pm \frac{\sqrt{3}}{2},
\]

\[
\theta = \pm \frac{\pi}{6}.
\]

So the only critical points in the interval are \( \theta = \frac{\pi}{6}, \theta = -\frac{\pi}{6} \).

(b) (2 pts) Classify each critical point from part (a) as a local minimum or a local maximum. Justify your answers.

Solution: Use the Second Derivative Test. Compute \( h''(\theta) = 2 \sec \theta (\sec \theta \tan \theta) = 2 \sec^2 \theta \tan \theta \). So \( h''(\frac{\pi}{6}) = 2 \sec^2 \frac{\pi}{6} \tan \frac{\pi}{6} = 2 \left( \frac{2}{\sqrt{3}} \right)^2 \cdot \frac{1}{\sqrt{3}} > 0 \), and so \( \frac{\pi}{6} \) is a local minimum. On the other hand, \( h''(-\frac{\pi}{6}) = 2 \sec^2 (-\frac{\pi}{6}) \tan (-\frac{\pi}{6}) = 2 \left( \frac{2}{\sqrt{3}} \right)^2 \cdot (-\frac{1}{\sqrt{3}}) < 0 \). So \( -\frac{\pi}{6} \) is a local maximum.

(c) (2 pts) Does \( h(\theta) \) have a global maximum point on the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\)? Why or why not?

Solution: \( h(\theta) \) does not have a global maximum point on this interval since

\[
\lim_{\theta \to -\frac{\pi}{2}} h(\theta) = \lim_{\theta \to -\frac{\pi}{2}} (3 \tan \theta - 4\theta) = 3(\infty) - 4\left( \frac{\pi}{2} \right) = \infty.
\]

So this infinite limit is larger than the value of \( h \) at the local max at \( \theta = -\frac{\pi}{6} \).