1. (4 pts) Use an appropriate linear approximation to approximate $\sqrt[3]{7.97}$. Your answer should be a decimal number. Show your work.

**Solution:** Let $f(x) = \sqrt[3]{x}$ and $a = 8$. Then compute $f(a) = \sqrt[3]{8} = 2$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ and $f'(a) = \frac{1}{3} \cdot 8^{-\frac{2}{3}} = \frac{1}{3} \cdot 2^{-2} = \frac{1}{3} \cdot \frac{1}{4}$. Then the linear approximation

$$L(x) = f(a) + f'(a)(x - a) = 2 + \frac{1}{3} \cdot \frac{1}{4}(x - 8),$$

and so

$$L(7.97) = 2 + \frac{1}{3} \cdot \frac{1}{4}(7.97 - 8) = 2 + \frac{1}{3} \cdot \frac{1}{4}(-0.03) = 2 - \frac{1}{4}(0.01) = 2 - 0.0025 = 1.9975$$

2. (6 pts) Compute the following limits. Each answer should be a real number, $+\infty$, $-\infty$, or “does not exist.” Show your work.

(a) $\lim_{x \to \infty} \frac{3}{x^2 + e^x}$.

**Solution:** As $x \to \infty$, $x^2 \to \infty$ and also $e^x \to \infty$. Thus the sum $x^2 + e^x \to \infty$ as well. So the quotient $\frac{3}{x^2 + e^x} \to 0$ as $x \to \infty$.

(b) $\lim_{t \to 0^+} \cot t$.

**Solution:** $\cot t = \frac{\cos t}{\sin t}$ and so if $t = 0$, we get the quotient $\frac{1}{0}$. So we expect the limit to be infinite, and we must determine the sign. As $t \to 0^+$, both $\sin t$ and $\cos t$ are positive, and so the limit $\lim_{t \to 0^+} \cot t = +\infty$.

3. (9 pts) Compute the following derivatives. Show your work.

(a) $\frac{d}{dt}(t^2 \ln t)$.

**Solution:** Use the Product Rule to compute the derivative as $2t \ln t + t^2 \cdot \frac{1}{t} = 2t \ln t + 1$.

(b) $\frac{d}{dx} \tan^{-1}(\frac{1}{x})$.

**Solution:** Use the Chain Rule to compute

$$\frac{d}{dx} \tan^{-1}(\frac{1}{x}) = \frac{1}{1 + (\frac{1}{x})^2} \cdot \left( -\frac{1}{x^2} \right) = -\frac{1}{x^2 + 1}.$$  

(c) $(f^{-1})'(0)$, where $f(x) = 2x + \sin x$.

**Solution:** Since $f(0) = 2 \cdot 0 + \sin 0 = 0$, we also have $f^{-1}(0) = f^{-1}(f(0)) = 0$. Also, $f'(x) = 2 + \cos x$. So compute

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = \frac{1}{2 + \cos 0} = \frac{1}{3}.$$
4. (8 pts) Consider a rectangular box with a square base, which will consist of 4 vertical sides, the base, and the top. If the surface area of the box is constrained to be 24 ft$^2$, what are the dimensions of the box with maximum volume? Show your work.

**Solution:** The volume of a box of dimensions $\ell, w, h$ is $V = \ell wh$, and the surface area is $A = 2\ell w + 2\ell h + 2wh$. For a square base, we have $\ell = w$, and so $V = w^2h$ and $A = 2w^2 + 4wh$.

The constraint is $24 = A = 2w^2 + 4wh$. Solving for $h$ gives

$$h = \frac{24 - 2w^2}{4w} = \frac{6}{w} - \frac{w}{2}.$$ 

The objective function

$$V = w^2h = w^2\left(\frac{6}{w} - \frac{w}{2}\right) = 6w - \frac{w^3}{2}.$$ 

The interval of $w$ follows from the conditions $h > 0$ and $w > 0$. The formula for $h$ in terms of $w$ shows that $h > 0$ when $w < 2\sqrt{3}$. So the interval for $w$ is $0 < w < 2\sqrt{3}$.

To maximize $V$, compute

$$\frac{dV}{dw} = 6 - \frac{3}{2}w^2$$

and the critical points are when $6 - \frac{3}{2}w^2 = 0$, $w^2 = 4$, $w = 2$ (we need only consider positive $w$). We can also compute for the endpoints of the interval $V(0) = V(2\sqrt{3}) = 0$. Thus $V(2) = 6 - 2 - \frac{2^3}{2} = 12 - 4 = 8$ is a global maximum. Thus $w = \ell = 2$ ft, and $h = \frac{6}{2} - \frac{2}{2} = 2$ ft also. The volume is 8 ft$^3$. 

5. (8 pts) On the axes provided below, sketch the graph of a function $y = f(x)$ which has the following properties:

- $f$ has domain consisting of the intervals $(-2, 0), (0, 2)$.
- For all $x$ in the domain, $f(-x) = -f(x)$.
- $f(1) = f(-1) = 0$.
- $f''(1) = f''(-1) = 0$, while $f'(x) > 0$ for all other $x$ in the domain.
- $\lim_{x \to 0^-} f(x) = \lim_{x \to 2^-} f(x) = \infty$.
- $\lim_{x \to 0^+} f(x) = \lim_{x \to 2^+} f(x) = -\infty$.
- $f''(x) < 0$ for $x$ in $(-2, -1)$ and $x$ in $(0, 1)$.
- $f''(x) > 0$ for $x$ in $(-1, 0)$ and $x$ in $(1, 2)$.
6. (8 pts) Consider the function \( g(x) = x - \ln x \).

(a) What is the domain of \( g(x) \)?

**Solution:** The domain of \( \ln x \) is \( x > 0 \), and so \( x > 0 \) is the domain of \( g(x) \) also.

(b) Find all the intervals on which \( g \) is increasing. Also find all intervals on which \( g \) is decreasing. Find all critical points of \( g \). Show your work.

**Solution:** Compute \( g'(x) = 1 - \frac{1}{x} \) and so \( g'(x) = 0 \) if \( 1 - \frac{1}{x} = 0 \), \( x - 1 = 0 \), \( x = 1 \). So \( x = 1 \) is the only critical point in the domain \((0, \infty)\). Now check the sign of \( g'(x) \) for \( x \) in the subintervals \((0, 1)\) and \((1, \infty)\): if \( x = \frac{1}{2} \), \( g'(\frac{1}{2}) = 1 - \frac{1}{\frac{1}{2}} = 1 - 2 < 0 \).

Therefore \( g'(x) < 0 \) for \( x \) in \((0, 1)\). \( g(x) \) is decreasing there. On the other hand for \( x = 2 \), \( g'(2) = \frac{1}{2} - \frac{1}{2} = \frac{1}{2} > 0 \), and so \( g'(x) > 0 \) for \( x \) in \((1, \infty)\). So \( g(x) \) is increasing there.

(c) Find all intervals on which \( g \) is concave up. Find all intervals on which \( g \) is concave down. Show your work.

**Solution:** Compute \( g''(x) = \frac{1}{x^2} \). This is always positive. So \( g(x) \) is concave up on its domain \((0, \infty)\).