1. (4 pts) Use an appropriate linear approximation to approximate $\sqrt{16.8}$. Your answer should be a decimal number. Show your work.

**Solution:** Let $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $a = 16$. Then compute $f(a) = \sqrt{16} = 2$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ and $f'(a) = \frac{1}{4} \cdot 16^{-\frac{3}{2}} = \frac{1}{4} \cdot 2^{-3} = \frac{1}{4} \cdot \frac{1}{8}$. Then the linear approximation

$$L(x) = f(a) + f'(a)(x - a) = 2 + \frac{1}{4} \cdot \frac{1}{8}(x - 16),$$

and so

$$L(16.8) = 2 + \frac{1}{4} \cdot \frac{1}{8}(16.8 - 16) = 2 + \frac{1}{4} \cdot \frac{1}{8}(0.8) = 2 + \frac{1}{4}(0.1) = 2 + 0.025 = 2.025$$

2. (6 pts) Compute the following limits. Each answer should be a real number, $+\infty$, $-\infty$, or “does not exist.” Show your work.

(a) $\lim_{x \to \infty} \frac{x^2}{\tan^{-1}x}$.

**Solution:** As $x \to \infty$, $x^2 \to \infty$, while $\tan^{-1}x \to \frac{\pi}{2}$. Thus the limit is $+\infty$.

(b) $\lim_{t \to 0^-} \cot t$.

**Solution:** $\cot t = \frac{\cos t}{\sin t}$ and so if $t = 0$, we get the quotient $\frac{1}{0}$. So we expect the limit to be infinite, and we must determine the sign. As $t \to 0^-$, both $\sin t < 0$, while $\cos t > 0$, and so the limit $\lim_{t \to 0^+} \cot t = -\infty$.

3. (9 pts) Compute the following derivatives. Show your work.

(a) $\frac{d}{dx} \cos^{-1}(\sqrt{x})$.

**Solution:** Use the Chain Rule to compute the derivative as

$$\frac{1}{\sqrt{1 - (\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{1 - x\sqrt{x}}} = \frac{1}{2\sqrt{x} - x^2}.$$

(b) $\frac{d}{dt} \ln t$.

**Solution:** Use the Quotient Rule to find

$$\frac{d}{dt} \ln t = \frac{t \cdot \frac{1}{t} - (\ln t)1}{t^2} = \frac{1 - \ln t}{t^2}.$$

(c) $(f^{-1})'(0)$, where $f(x) = \sin x - 3x$.

**Solution:** Since $f(0) = \sin 0 - 3 \cdot 0 = 0$, we also have $f^{-1}(0) = f^{-1}(f(0)) = 0$. Also, $f'(x) = \cos x - 3$. So compute

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = \frac{1}{\cos 0 - 3} = -\frac{1}{2}.$$
4. (8 pts) Consider a rectangular box with a square base, which will consist of 4 vertical sides, the base, and the top. If the surface area of the box is constrained to be 6 \( \text{ft}^2 \), what are the dimensions of the box with maximum volume? Show your work.

**Solution:** The volume of a box of dimensions \( \ell, w, h \) is \( V = \ell w h \), and the surface area is \( A = 2\ell w + 2\ell h + 2wh \). For a square base, we have \( \ell = w \), and so \( V = w^2h \) and \( A = 2w^2 + 4wh \).

The constraint is \( 6 = A = 2w^2 + 4wh \). Solving for \( h \) gives

\[
h = \frac{6 - 2w^2}{4w} = \frac{3}{2w} - \frac{w}{2}.
\]

The objective function

\[
V = w^2h = w^2\left(\frac{3}{2w} - \frac{w}{2}\right) = \frac{3}{2}w - \frac{w^3}{2}.
\]

The interval of \( w \) follows from the conditions \( h > 0 \) and \( w > 0 \). The formula for \( h \) in terms of \( w \) shows that \( h > 0 \) when \( w < \sqrt{3} \). So the interval for \( w \) is \( 0 < w < \sqrt{3} \).

To maximize \( V \), compute

\[
\frac{dV}{dw} = \frac{3}{2} - \frac{3}{2}w^2
\]

and the critical points are when \( \frac{3}{2} - \frac{3}{2}w^2 = 0 \), \( w^2 = 1 \), \( w = 1 \) (we need only consider positive \( w \)). We can also compute for the endpoints of the interval \( V(0) = V(\sqrt{3}) = 0 \). Thus \( V(1) = \frac{3}{2} \cdot 1 - \frac{1}{2} \cdot 1^3 = \frac{3}{2} - \frac{1}{2} = 1 \) is a global maximum. Thus \( w = \ell = 1 \text{ ft} \), and \( h = \frac{3}{2} - \frac{1}{2} = 1 \text{ ft} \) also. The volume is 1 \( \text{ft}^3 \).
5. (8 pts) On the axes provided below, sketch the graph of a function \( y = f(x) \) which has the following properties:

- \( f \) is continuous on its domain \((-\infty, 2)\).
- \( \lim_{x \to -\infty} f(x) = \infty \).
- \( f(0) = 1, \ f(1) = 0, \ f(1.5) = 2, \ f'(1) \) does not exist.
- \( f'(x) > 0 \) for \( x \) in \((1, 2)\).
- \( f'(x) < 0 \) for \( x < 1 \).
- \( f''(x) < 0 \) for \( x \) in \((-\infty, 1)\) and \( x \) in \((1, 1.5)\).
- \( f''(x) > 0 \) for \( x \) in \((1.5, 2)\).
- \( \lim_{x \to -\infty} f(x) = 2. \)
6. (8 pts) Consider the function \( g(x) = \ln x - 2x \).

(a) What is the domain of \( g(x) \)?

**Solution:** The domain of \( \ln x \) is \( x > 0 \), and so \( x > 0 \) is the domain of \( g(x) \) also.

(b) Find all the intervals on which \( g \) is increasing. Also find all intervals on which \( g \) is decreasing. Find all critical points of \( g \). Show your work.

**Solution:** Compute \( g'(x) = \frac{1}{x} - 2 \) and so \( g'(x) = 0 \) if \( \frac{1}{x} - 2 = 0 \), \( 1 - 2x = 0 \), \( x = \frac{1}{2} \).

So \( x = \frac{1}{2} \) is the only critical point in the domain \((0, \infty)\). Now check the sign of \( g'(x) \) for \( x \) in the subintervals \((0, \frac{1}{2})\) and \((\frac{1}{2}, \infty)\): if \( x = \frac{1}{4} \), \( g'(\frac{1}{4}) = \frac{4}{4} - 2 = 4 - 2 > 0 \).

Therefore \( g'(x) > 0 \) for \( x \) in \((0, \frac{1}{2})\). \( g(x) \) is increasing there. On the other hand for \( x = 1 \), \( g'(1) = \frac{1}{1} - 2 = -1 < 0 \), and so \( g'(x) \) \( < 0 \) for \( x \) in \((\frac{1}{2}, \infty)\). So \( g(x) \) is decreasing there.

(c) Find all intervals on which \( g \) is concave up. Find all intervals on which \( g \) is concave down. Show your work.

**Solution:** Compute \( g''(x) = -\frac{1}{x^2} \). This is always negative. So \( g(x) \) is concave down on its domain \((0, \infty)\).