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Symmetry Breaking and Other Phenomena in the Optimization of Eigenvalues for Composite Membranes

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Abstract: We consider the following eigenvalue optimization problem: Given a bounded domain $\Omega \subset \mathbb{R}$ and numbers $\alpha > 0$, $A \in [0, |\Omega|]$, find a subset $D \subset \Omega$ of area A for which the first Dirichlet eigenvalue of the operator $-\Delta + \alpha\chi_D$ is as small as possible.

We prove existence of solutions and investigate their qualitative properties. For example, we show that for some symmetric domains (thin annuli and dumbbells with narrow handle) optimal solutions must possess fewer symmetries than Ω; on the other hand, for convex Ω reflection symmetries are preserved.

Also, we present numerical results and formulate some conjectures suggested by them.

1. Problem and Main Results

We study qualitative properties of solutions of a certain eigenvalue optimization problem. In physical terms, the problem can be stated as follows:

Problem (P). *Build a body of prescribed shape out of given materials (of varying densities) in such a way that the body has a prescribed mass and so that the basic frequency of the resulting membrane (with fixed boundary) is as small as possible.*

In fact, we will consider a more general problem, which we now state in mathematical terms: Given a domain $\Omega \subset \mathbb{R}^n$ (bounded, connected, with Lipschitz boundary) and numbers $\alpha > 0$, $A \in [0, |\Omega|]$ (with $|\cdot|$ denoting volume). For any measurable subset $D \subset \Omega$ let χ_D be its characteristic function and $\lambda_\Omega(\alpha, D)$ the lowest eigenvalue λ of the problem

$$\begin{aligned} -\Delta u + \alpha\chi_D u &= \lambda u \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Define

$$\Lambda_\Omega(\alpha, A) = \inf_{\substack{D\subset\Omega \\ |D|=A}} \lambda_\Omega(\alpha, D). \tag{2}$$

Any minimizer D in (2) will be called an *optimal configuration* for the data (Ω, α, A). If D is an optimal configuration and u satisfies (1) then (u, D) will be called an *optimal pair* (or *solution*). Our problem now reads:

Problem (M). *Study existence, uniqueness and qualitative properties of optimal pairs.*

As is well-known, u is uniquely determined, up to a scalar multiple, by D, and may be chosen to be positive on Ω. In addition, we will always assume

$$\int_\Omega u^2 = 1.$$

(Integrals over Ω are always taken with respect to the standard measure.) Clearly, changing D by a set of measure zero does not affect $\lambda_\Omega(\alpha, D)$ or u. Therefore, we will consider sets D that differ by a null-set as equal.

At first sight, it is not obvious that problem (M) generalizes problem (P). In fact, we will see (Theorem 13) that there is a number $\overline{\alpha}_\Omega(A) > 0$ such that solutions of problem (P) are in one to one correspondence with solutions of problem (M) with parameters in the range $\alpha \leq \overline{\alpha}_\Omega(A)$. The number $\overline{\alpha}_\Omega(A)$ is characterized as the unique value of α satisfying

$$\Lambda_\Omega(\overline{\alpha}_\Omega(A), A) = \overline{\alpha}_\Omega(A), \tag{3}$$

see Proposition 10.

Our investigations are theoretical and numerical: Numerical results (obtained by M.I. and I.O.) suggest properties of optimal configurations; this leads to the formulation of conjectures, and some of these are proved rigorously (by S.C., D.G. and K.K.).

A central tool in our investigations is the variational characterization of the eigenvalue:

$$\lambda_\Omega(\alpha, D) = \inf_{u\in H_0^1(\Omega)} R_\Omega(u, \alpha, D), \quad R_\Omega(u, \alpha, D) := \frac{\int_\Omega |\nabla u|^2 + \alpha \int_\Omega \chi_D u^2}{\int_\Omega u^2},$$

and the eigenfunction u is a minimizer. So $\Lambda_\Omega(\alpha, A)$ is characterized by

$$\Lambda_\Omega(\alpha, A) = \inf_{\substack{u\in H_0^1(\Omega) \\ |D|=A}} R_\Omega(u, \alpha, D).$$

We first prove the following theorem on existence and basic properties of solutions. It is fundamental for all further considerations.

Theorem 1. *For any $\alpha > 0$ and $A \in [0, |\Omega|]$ there exists an optimal pair. Moreover, any optimal pair (u, D) has the following properties:*

(a) *$u \in C^{1,\delta}(\Omega) \cap H^2(\Omega) \cap C^\gamma(\overline{\Omega})$ for some $\gamma > 0$ and every $\delta < 1$.*
(b) *D is a sublevel set of u, i.e. there is a number $t \geq 0$ such that*

$$D = \{u \leq t\}.$$

(c) *Every level set* $\{u = s\}$, $s \geq 0$, *has measure zero, except possibly in the case* $\alpha = \overline{\alpha}_\Omega(A)$, $s = t$.

Here we use the short notation $\{u = t\} = \{x : u(x) = t\}$. Since χ_D is discontinuous, solutions u may not be twice differentiable, so Eq. (1) is understood in the weak sense.

Note that Theorem 1(b) shows in particular that our problem is equivalent to finding the smallest eigenvalue and associated eigenfunctions of the nonlinear problem (with free variables u and t)

$$\begin{aligned} -\Delta u + \alpha\chi_{\{u\leq t\}} u &= \lambda u \quad \text{ on } \Omega, \\ u &= 0 \quad \text{ on } \partial\Omega, \\ |\{u \leq t\}| &= A. \end{aligned} \tag{4}$$

The question of *uniqueness* is much more subtle: For some domains Ω there will be a unique optimal pair for all α, A, while for others there will be many, for certain ranges of α, A. This follows from our results on symmetry preservation and symmetry breaking below.

We now list a few questions that naturally come to mind:

(SY) If Ω has symmetries, does D have the same symmetries? (Note that if Ω and D have a symmetry in common then u will also have this symmetry since it is uniquely determined by Ω and D.)
(CX) Assume Ω is convex. Is $D^c := \Omega \setminus D$ convex? Is D unique?
(CN) Is D or D^c connected?
(FB) What is the regularity of the *free boundary* ∂D?

We give partial answers to all of these questions. Some proofs, mainly relating to (FB), and additional results can be found in the companion paper [CGK]. Many open problems remain, see Sect. 6.

At this point, the reader is invited to look at Figs. 1–3 for a first impression.

We now state our qualitative results. As a general convention, constants only depend on the quantities indicated as subscripts or in parentheses, unless otherwise specified. Often we suppress the subscript Ω.

First, as an easy consequence of Theorem 1 one has:

Theorem 2. *Fix* $\alpha > 0$, $A > 0$, *and let* D *be an optimal configuration.*

(a) *D contains a tubular neighborhood of the boundary* $\partial\Omega$.
(b) *If* $\alpha < \overline{\alpha}_\Omega(A)$ *then every connected component* D_0 *of the interior of* D *hits the boundary, i.e.* $\overline{D_0} \cap \partial\Omega \neq \emptyset$.

In particular, if Ω *is simply connected and* $\alpha < \overline{\alpha}_\Omega(A)$ *then* D *is connected.*

The number $\overline{\alpha}_\Omega(A)$ was defined above, see (3). The significance of the condition $\alpha < \overline{\alpha}_\Omega(A)$ is that it is equivalent to $\Delta u < 0$ on Ω. One always has

$$\overline{\alpha}_\Omega(A) \geq \mu_\Omega.$$

Here and throughout the paper, μ_Ω denotes the first eigenvalue of the Dirichlet Laplacian on Ω, and ψ_Ω the positive, L^2-normalized eigenfunction:

$$-\Delta\psi_\Omega = \mu_\Omega\psi_\Omega \quad \text{on } \Omega, \quad \psi_\Omega = 0 \quad \text{on } \partial\Omega.$$

Next, we consider the dependence of Λ_Ω and solutions (u, D) on α and A. Here it is convenient to formulate our problem also for $\alpha = 0$, as follows: If $\alpha = 0$ then a solution (unique in this case) is a pair (ψ_Ω, D), where D is the sublevel set of ψ_Ω of area A. (Since ψ_Ω is real analytic and non-constant, such D exists for every A and is unique.)

We will prove strict monotonicity and Lipschitz continuity of Λ_Ω in both parameters (Prop. 10). Continuous dependence of optimal pairs (u, D) on the parameters may be expected only at parameter values where they are unique. This is the case, in particular, if $\alpha = 0$ or $A = 0$ or $A = |\Omega|$; in these cases $u = \psi_\Omega$, and the continuity is proved in [CGK]. Here we only state the results. They are used only in the proof of Theorem 9. For example, we have the following:

Theorem 3. *For $s \geq 0$ let $[\Omega]^s = \{\psi_\Omega \leq s\}$, where ψ_Ω is the positive L^2-normalized first eigenfunction of $-\Delta$ on Ω. Fix $A \in [0, |\Omega|]$ and choose t_Ω such that $|[\Omega]^{t_\Omega}| = A$. Then for any $\delta > 0$ there is $\alpha_0 = \alpha_0(\delta, \Omega)$ such that whenever $\alpha < \alpha_0$ and D is an optimal configuration for (α, A) then $|t - t_\Omega| < \delta$ and*

$$[\Omega]^{t_\Omega - \delta} \subset D \subset [\Omega]^{t_\Omega + \delta}.$$

We now address questions of *symmetry*. First, we prove *symmetry preservation* in the presence of convexity:

Theorem 4. *Assume that the domain Ω is symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$. In other words, for each $x' = (x_2, \dots, x_n)$ the set*

$$\{x_1 : (x_1, x') \in \Omega\} \tag{5}$$

is either empty or an interval of the form $(-c, c)$.

Then for any solution (u, D) both u and D are symmetric with respect to $\{x_1 = 0\}$, D^c is convex with respect to $\{x_1 = 0\}$, and u is decreasing in x_1 for $x_1 \geq 0$.

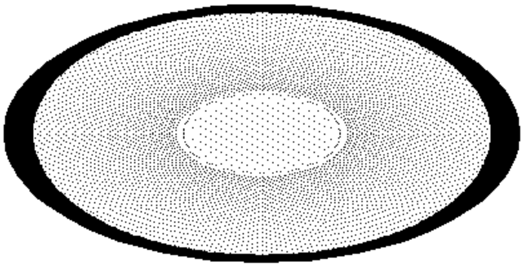


Fig. 1. Ellipse (with $\alpha = 1$, $A = 1$, $|\Omega| = 6.3$). Optimal configuration D in black

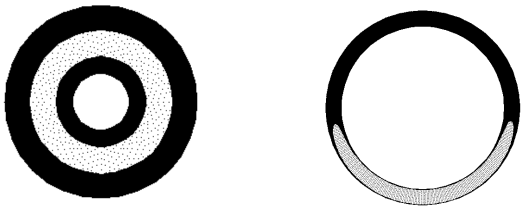
For example, any solution in an elliptic region has a double reflection symmetry, see Fig. 1. The principal tool here is Steiner symmetrization. See [K2] for an overview on such methods. Theorem 4 easily implies the following uniqueness result (the only case where we can prove uniqueness!):

Corollary 5. *Let $\Omega = \{|x| < 1\}$ be the ball. Then there is a unique optimal configuration D for any α, A, and D is a shell region*

$$D = \{x : r(A) < |x| < 1\}.$$

One of the most interesting phenomena studied in this paper is *symmetry breaking* for certain plane domains Ω. That is, an optimal configuration D may have less symmetry than Ω. We will prove it for two types of domains: Thin annuli and dumbbells with narrow handle. An annulus has rotational symmetry, a dumbbell has a reflection symmetry.

Theorem 6. *Fix* $\alpha > 0$ *and* $\delta \in (0, 1)$. *For* $a > 0$ *let* $\Omega_a = \{x \in \mathbb{R}^2 : a < |x| < a+1\}$. *There exists* $a_0 = a_0(\alpha, \delta)$ *such that whenever* $a > a_0$ *and* D *is an optimal configuration for* Ω_a *with parameters* α *and* $A = \delta|\Omega_a|$ *then* D *is not rotationally symmetric.*



a $\alpha = 1, \delta = 0.64, \tau = 3.5$ **b** Thinner annulus ($\tau = 1.2$)

c Stronger non-linearity ($\alpha = 10$), **d** Larger area of D ($\delta = 0.83$)

Fig. 2. Symmetry breaking on annuli: The parameters are α (the "strength" of the non-linearity), $\delta = A/|\Omega|$ (the relative size of D), and $\tau = r_{\text{out}}/r_{\text{in}}$ (the ratio of outer and inner radius). In each of **b**, **c**, **d** only one parameter is changed, compared with **a**. Optimal configuration D in black

See Fig. 2a and b. For dumbbells we prove a little more than symmetry breaking:

Theorem 7. *For* $h \in (0, 1)$ *define the dumbbell with handle width* $2h$,

$$\Omega_h = B_1(-2, 0) \cup ((-2, 2) \times (-h, h)) \cup B_1(2, 0),$$

where $B_r(p) = \{x \in \mathbb{R}^2 : |x - p| < r\}$. *Fix* $\alpha > 0$ *and* $A \in (0, 2\pi)$. *Then there is* $h_0 = h_0(\alpha, A) > 0$ *such that we have for* $h < h_0$:

(a) *Any optimal pair* (u, D) *is not symmetric with respect to the* x_2*-axis.*
(b) *If* $A > \pi$ *then for any optimal pair* (u, D) *the complement* D^c *is contained in one of the lobes (i.e. one of the balls* $B_1(\pm 2, 0)$*).*

See Fig. 3 for part (a). In fact, similar results hold for more general dumbbells.

As we remarked before, symmetry breaking implies non-uniqueness: For example for a dumbbell the pair (u', D') obtained from a solution (u, D) by reflection in the x_2-axis will be a solution, and different from (u, D) by the theorem.

The following result on the regularity of the free boundary is proved in [CGK]:

Theorem 8. *If* (u, D) *is an optimal pair,* $x \in \partial D$ *and* $\nabla u(x) \neq 0$ *then* ∂D *is a real analytic hypersurface near* x.

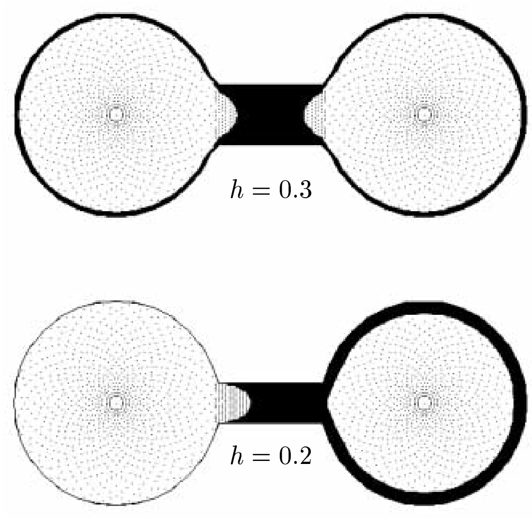


Fig. 3. Symmetry breaking on dumbbells: The parameters are $\alpha = 0.1$, $A = 1$. Symmetry breaking occurs when the width of the handle $(2h)$ is decreased. The "lobes" are unit circles with centers 4 units apart. Optimal configuration D in black

The difficulty is that χ_D is discontinuous at $x \in \partial D$, so u is not even C^2 there. That the level set $\{u = t\}$ has C^ω regularity nevertheless is proved by introduction of suitable local coordinates (with u as one coordinate) and analysis of the resulting nonlinear elliptic equation.

Similar arguments and continuity considerations for α near zero allow us to give partial answers to problems (CX) and (FB):

Theorem 9. *Suppose Ω is convex and has a C^2 boundary. Then there is $\alpha_0(A, \Omega) > 0$ such that for any $\alpha < \alpha_0$ and any optimal configuration D, one has:*

(a) *$\partial D \cap \Omega$ is real analytic;*
(b) *D^c is convex.*

Problem (P) and generalizations of it (to higher eigenvalues and to a maximization problem), but with fewer qualitative results, were studied before in [Kr, CM], and [C] (where Theorem 4 is stated, but the proof is incomplete since the case of equality in the rearrangement inequalities is not addressed).

Problems similar to problem (M) (e.g. with L^p potentials) were considered in [AH, Eg, AHS, CL], and [HKK].

The paper is organized as follows: In Sect. 2, we prove Theorems 1 and 2 and discuss the parameter dependence of Λ_Ω. Also, in Subsect. 2.3 we discuss the relation of problems (P) and (M). In Sect. 3 we prove Theorems 4, 6, and 7 on symmetry questions, and Corollary 5. In Sect. 4 we prove Theorem 9. In Sect. 5 we describe the numerical algorithm used. In Sect. 6 we state some open problems and conjectures. Finally, we collect some standard facts about elliptic PDEs in the Appendix.

2. Basic Results

2.1. Existence and regularity. Proof of Theorem 1. We first prove *existence and regularity:* The regularity statements in (a) hold for solutions of equations

$$-\Delta u + \rho u = 0$$

with ρ bounded by standard elliptic theory, see for example [GT, Theorem 8.29 and Corollary 8.36].

To prove existence, fix α and A, and write $\Lambda = \Lambda_\Omega(\alpha, A)$, $\lambda(D) = \lambda_\Omega(\alpha, D)$ for simplicity. Let D_j be a minimizing sequence, i.e. $\lambda(D_j) \to \Lambda$ as $j \to \infty$. Let $u_j \in H_0^1$ (all function spaces are defined on Ω) be the positive L^2-normalized first eigenfunction of $-\Delta + \alpha\chi_{D_j}$. Since $\lambda(D_j)$ is bounded, the sequence $\{u_j\}$ is bounded in H_0^1. Also, $\{\chi_{D_j}\}$ is bounded in L^2. Therefore, we may choose a subsequence (again denoted u_j, D_j) and $u \in H_0^1$, $\eta \in L^2$ such that $u_j \rightharpoonup u$ in H_0^1 (weak convergence) and $\chi_{D_j} \rightharpoonup \eta$ in L^2. This implies $u_j \to u$ (strongly) in L^2, $\chi_{D_j} u_j \rightharpoonup \eta u$ in L^2, and $\int_\Omega \eta = A$. Now taking limits in the weak form of the eigenvalue equation

$$\int_\Omega \nabla u_j \cdot \nabla \psi + \alpha \int_\Omega \chi_{D_j} u_j \psi = \lambda(D_j) \int_\Omega u_j \psi \quad \forall \psi \in H_0^1$$

we get

$$-\Delta u + \alpha \eta u = \Lambda u \quad \text{(weakly)}. \tag{6}$$

We have

$$0 \le \eta \le 1 \quad \text{a.e.},$$

since $0 \le \chi_{D_j} \le 1$ for all j and weak convergence preserves pointwise inequalities a.e. (exercise!). Therefore, u has the regularity stated in (a).

It remains to prove that η may be replaced by a characteristic function. Since $\int_\Omega u^2 = 1$, (6) shows that

$$\int_\Omega |\nabla u|^2 + \alpha \int_\Omega \eta u^2 = \Lambda. \tag{7}$$

Now the minimization problem

$$\inf_{\substack{\eta : \int \eta = A \\ 0 \le \eta \le 1}} \int_\Omega \eta u^2$$

has a solution $\eta = \chi_D$, where D is any set with $|D| = A$ and

$$\{u < t\} \subset D \subset \{u \le t\}, \quad t := \sup\{s : |\{u < s\}| < A\} \tag{8}$$

(compare the "bathtub principle", Theorem 1.18 in [LL]). Therefore, we get from (7)

$$\int_\Omega |\nabla u|^2 + \alpha \int_\Omega \chi_D u^2 \le \Lambda.$$

By definition of Λ as a minimum, this must actually be an equality, and (u, D) is a solution.

(b) Let (u, D) be any solution. Then it is obvious that (8) must hold (always up to a set of measure zero; if (8) didn't hold then one could reduce $\int_D u^2$ by shifting a part of D from $\{u > t\}$ to $\{u \leq t\}$). Set $\mathcal{N}_s = \{u = s\}$ for any $s > 0$. Using Lemma 7.7 from [GT] twice, we see that $\Delta u = 0$ a.e. on $\mathcal{N}_s$ (since $u \equiv \text{const}$ on $\mathcal{N}_s$; recall that u is in H^2). Therefore,

$$(\Lambda - \alpha\chi_D)u = 0 \quad \text{a.e. on } \mathcal{N}_s. \tag{9}$$

Since $u > 0$ and $\Lambda > 0$, this shows that $D^c \cap \mathcal{N}_s$ has measure zero. Taking $s = t$ we get (b).

(c) If $s > t$ then $\mathcal{N}_s \subset D^c$, so $|\mathcal{N}_s| = 0$ by (9). The same argument works if $s = t$ and $\alpha \neq \Lambda$.

Finally, u satisfies $-\Delta u = (\Lambda - \alpha)u$ on the open set $\{u < t\}$, hence u is real analytic there, and therefore the level sets $\mathcal{N}_s$ have measure zero for $s < t$. □

Proof of Theorem 2. Part (a) is clear from Theorem 1(b). To prove (b), assume this was false. Then there is an open subset $D_0 \subset \{u \leq t\}$ with $\partial D_0 \subset \overline{D^c} = \{u \geq t\}$ and therefore $u = t$ on ∂D_0. Then u assumes a minimum at some $x_0 \in D_0$. But this is a contradiction since $\alpha < \overline{\alpha}_\Omega(A)$ implies $\Lambda(\alpha, A) > \alpha$ (see Proposition 10 below) and therefore $\Delta u = (\alpha - \Lambda(\alpha, A))u < 0$ on D_0. □

2.2. Parameter dependence of Λ.

Proposition 10. (a) *The function* $(\alpha, A) \mapsto \Lambda(\alpha, A)$ *is Lipschitz continuous, uniformly on bounded sets. More precisely, we have, for any* $\alpha, \alpha' \geq 0$, $A, A' \in [0, |\Omega|]$,

$$|\Lambda(\alpha, A) - \Lambda(\alpha', A')| \leq \\ |\alpha - \alpha'| \frac{\max\{A, A'\}}{|\Omega|} + |A - A'| \min\{\alpha, \alpha'\} C_{\Omega, \max\{\alpha, \alpha'\}} \tag{10}$$

with $C_{\Omega,\alpha}$ *bounded for* α *bounded.*

(b) $\Lambda(\alpha, A)$ *is strictly increasing in* A *for fixed* $\alpha > 0$, *strictly increasing in* α *for fixed* $A > 0$, *and* $\Lambda(\alpha, A) - \alpha$ *is strictly decreasing in* α *for fixed* $A < |\Omega|$.

(c) *If* $A < |\Omega|$ *then there is a unique value* $\alpha = \overline{\alpha}_\Omega(A)$ *with*

$$\Lambda(\overline{\alpha}_\Omega(A), A) = \overline{\alpha}_\Omega(A). \tag{11}$$

The function $\overline{\alpha}_\Omega$ *is continuous and strictly increasing,* $\overline{\alpha}_\Omega(0) = \mu_\Omega$ *and* $\overline{\alpha}_\Omega(A) \to \infty$ *as* $A \to |\Omega|$.

Proof. (a) Write $\Lambda = \Lambda(\alpha, A)$ and $\Lambda' = \Lambda(\alpha', A')$, and let (u, D), (u', D') be minimizers for Λ, Λ' respectively. We may assume $\int_\Omega u^2 = \int_\Omega (u')^2 = 1$, so that

$$\Lambda = \int_\Omega |\nabla u|^2 + \alpha \int_D u^2, \quad |D| = A,$$

and similarly for Λ', etc. By symmetry of (10) we may assume that $A' \geq A$. Choose $D_1 \subset D'$ with $|D_1| = A$ and $D_1' \supset D$ with $|D_1'| = A'$. Here we may assume that D_1' is of the form $\{u \leq s\}$ for a suitable number s. Using the optimality of (u, D) for Λ we get

$$\Lambda \leq \int_\Omega |\nabla u'|^2 + \alpha \int_{D_1} (u')^2 = \Lambda' + (\alpha - \alpha') \int_{D'} (u')^2 - \alpha \int_{D' \setminus D_1} (u')^2. \tag{12}$$

Similarly, using the optimality of (u', D') for Λ' we get

$$\Lambda' \le \int_\Omega |\nabla u|^2 + \alpha' \int_{D_1'} u^2 = \Lambda + (\alpha' - \alpha) \int_{D_1'} u^2 + \alpha \int_{D_1' \setminus D} u^2. \tag{13}$$

Alternatively, we may rewrite this as

$$\Lambda' \le \Lambda + (\alpha' - \alpha) \int_D u^2 + \alpha' \int_{D_1' \setminus D} u^2. \tag{13'}$$

In order to estimate the integrals in (12), (13) and (13') which are multiplied by $\pm(\alpha - \alpha')$, observe that for any $s > 0$ and any function u we have

$$\frac{\int_{\{u \le s\}} u^2}{\int_\Omega u^2} \le \frac{|\{u \le s\}|}{|\Omega|}.$$

The other integrals are estimated using the uniform estimate (47): u solves the equation $-\Delta u + \alpha \chi_D u = \Lambda u$. Λ is bounded in terms of Ω and α since one may apply (12) with $\alpha' = 0$, $A = A'$, to obtain $\Lambda \le \mu_\Omega + \alpha$. Therefore, the uniform bound (47), applied to $G = \Omega$, yields

$$\int_{D_1' \setminus D} u^2 \le (A' - A) \sup_\Omega u^2 \le (A' - A) C_{\Omega,\alpha}.$$

Finally, we obtain (10) by applying these estimates to (12) and (13) in the case $\alpha \le \alpha'$, and to (12) and (13') if $\alpha \ge \alpha'$.

(b) This follows immediately from (12) and the unique continuation theorem.

(c) This follows easily from (a) and (b) since $\Lambda(\alpha, A) - \alpha$ equals $\mu_\Omega > 0$ for $\alpha = 0$ and tends to $-\infty$ as $\alpha \to \infty$ by (a). □

We now consider continuous dependence of optimal pairs (u, D) on the data. First, near $\alpha = 0$:

Proposition 11. *Fix $D \subset \Omega$. Let $u_{\alpha,D}$ be the (positive, L^2-normalized) first eigenfunction of $-\Delta + \alpha \chi_D$, and $\psi_\Omega = u_{0,D}$ the first eigenfunction of $-\Delta$. Then there is a constant $C = C_\Omega$ such that, for $0 \le \alpha \le 1$,*

$$\|u_{\alpha,D} - \psi_\Omega\| \le C\alpha,$$

in the $H^2(\Omega)$ and $L^\infty(\Omega)$ norms, and in $C^{1,\delta}(\Omega)$ if $\partial\Omega$ is in $C^{1,\delta}$.

Proof. See [CGK]. □

Proof of Theorem 3. This is almost immediate from Proposition 11, see [CGK]. □

Similarly, one has continuity in A at $A = 0$ and at $A = |\Omega|$. Here we only consider the latter case:

Proposition 12. *Let Ω be a smooth bounded domain and fix $\alpha > 0$. Let*

$$M = \max_\Omega \psi_\Omega.$$

Then, for any $\delta > 0$ there is $A_0 = A_0(\delta, \alpha, \Omega) < |\Omega|$ such that whenever $A > A_0$ and D is an optimal configuration for (α, A) then

$$D^c \subset \{\psi_\Omega > M - \delta\}.$$

Proof. See [CGK]. □

2.3. Relation of problems (P) and (M). We want to show that problem (P) (see Sect. 1) is a special case of problem (M).

The mathematical formulation of problem (P) is: Given $0 \le h < H$ (lower and upper bounds for the densities of the materials that are available) and the prescribed total mass $M \in [h|\Omega|, H|\Omega|]$, $M > 0$, consider measurable "density functions" ρ satisfying

$$h \le \rho \le H, \qquad \int_\Omega \rho = M.$$

Then the objective is to find ρ and u which realize the minimum in

$$\Theta(h, H, M) := \inf_\rho \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega \rho u^2}. \tag{14}$$

The corresponding eigenvalue problem is

$$-\Delta u = \Theta \rho u, \quad u_{|\partial\Omega} = 0. \tag{15}$$

(We assume the modulus of elasticity to be the same for all materials.) Problem (P) and problem (M) are related in the following way:

Theorem 13. (a) *If* (u, ρ) *is a minimizer for problem (P) then* ρ *is of the form*

$$\rho_D = h\chi_D + H\chi_{D^c}$$

for a set D *of the form* $D = \{u \le t\}$. *That is, only two types of materials occur.*

(b) *The pair* (u, ρ_D) *is a minimizer for problem (P), with parameter values* (h, H, M), *if and only if* (u, D) *is a minimizer (optimal pair) for problem (M), with parameter values* (α, A) *given by*

$$\alpha = (H - h)\Theta(h, H, M), \tag{16}$$

$$A = \frac{H|\Omega| - M}{H - h}. \tag{17}$$

The minimal eigenvalues are related by

$$\Lambda(\alpha, A) = H\Theta(h, H, M). \tag{18}$$

(c) *The values of* (α, A) *that occur when* h, H, M *vary are precisely those satisfying*

$$\begin{aligned} &A \in [0, |\Omega|), && 0 < \alpha \le \overline{\alpha}_\Omega(A) \quad \textit{or} \\ &A = |\Omega|, && 0 < \alpha < \infty, \end{aligned}$$

where $\overline{\alpha}_\Omega(A)$ *is defined in* (11). *In particular,* $\alpha = \overline{\alpha}_\Omega(A)$ *corresponds to* $h = 0$.

Note that problem (P) really depends on two parameters only since for $\kappa > 0$ one has

$$\Theta(\kappa h, \kappa H, \kappa M) = \kappa^{-1}\Theta(h, H, M),$$

with the same minimizers (up to a factor κ for ρ). This is obvious from (14).

Proof. (a) This is almost obvious from (14), and proved just like part (b) of Theorem 1.

(b) First, if $\rho = \rho_D$ and $|D| = A$ then $M = \int_\Omega \rho = Ah + (|\Omega| - A)H$, which gives (17).

Simple manipulation shows that

$$-\Delta u = \Theta\rho_D u = \Theta(h\chi_D + H\chi_{D^c})u \tag{19}$$

is equivalent to

$$-\Delta u + (H - h)\Theta\chi_D u = H\Theta u. \tag{20}$$

Now if (u, ρ_D) is a minimizer for problem (P) then it satisfies (19) with $\Theta = \Theta(h, H, M)$, and then (20) shows that $\Lambda(\alpha, A) \leq H\Theta(h, H, M)$ with α satisfying (16).

Conversely, if (u, D) is a minimizer for problem (M) with parameter values (α, A) given by (16), (17) then (20) holds with $H\Theta$ replaced by $\Lambda = \Lambda(\alpha, A)$, so instead of (19) we get $-\Delta u = \Theta\rho_D u + (\Lambda - H\Theta)u$, where $\Theta = \Theta(h, H, M)$. Multiplying by u and integrating gives

$$\int_\Omega |\nabla u|^2 = \Theta \int_\Omega \rho_D u^2 + (\Lambda - H\Theta)\int_\Omega u^2.$$

Now the definition of Θ implies that $\int_\Omega |\nabla u|^2 \geq \Theta \int_\Omega \rho_D u^2$, so we get $\Lambda \geq H\Theta$. This proves $\Lambda(\alpha, A) = H\Theta(h, H, M)$ and part (b).

(c) If $A = |\Omega|$ then $D = \Omega$, $\rho \equiv h$ and therefore $h\Theta(h, H, M) = \mu_\Omega$ from (15), so $\alpha = \frac{H-h}{h}\mu_\Omega$ can take any positive value by suitable choice of h and H.

Now let $A < |\Omega|$. By Prop. 10(b) and (c), α varies in the indicated range precisely when $\Lambda(\alpha, A) - \alpha$ varies in $[0, \mu_\Omega)$. From (16) and (18) one has

$$\Lambda(\alpha, A) - \alpha = h\Theta := h\Theta(h, H, M),$$

so we only need to show that $h\Theta$ has range $[0, \mu_\Omega)$ (with A fixed). First, $h\Theta \geq 0$ by definition, and $h\Theta = \Lambda - \alpha < \mu_\Omega$ by Prop. 10, since $\alpha = (H - h)\Theta > 0$, so the range of $h\Theta$ is contained in $[0, \mu_\Omega)$. Next, $h\Theta = 0$ for $h = 0$ (and then M can be adjusted to A), and in the limit $H = h$ one has $\rho \equiv h$ and $h\Theta = \mu_\Omega$, so when $H \to h$ then $h\Theta \to \mu_\Omega$, and clearly M can be adjusted to A. Using continuity of $h\Theta$ (which is proved as for Λ in Prop. 10) we get the claim. □

3. Symmetry Preservation and Symmetry Breaking

3.1. Symmetry preservation in the presence of convexity. Here we prove Theorem 4.

Proof of Theorem 4. We use Steiner symmetrization (symmetrically decreasing rearrangement) $u \mapsto u^\#$ with respect to the hyperplane $\{x_1 = 0\}$. This is defined as follows. Assume $u \in H_0^1(\Omega) \cap C^0(\Omega)$: For each x', $u^\#(\cdot, x')$ is the unique function of x_1 which is symmetric in x_1 and decreasing for $x_1 \geq 0$ such that $|\{x_1 : u^\#(x_1, x') > t\}| = |\{x_1 : u(x_1, x') > t\}|$ for all $t \in \mathbb{R}$. It is well-known (see, e.g., [LL, AB]) that, for all x' and $i = 1, \ldots, n$, with integrals taken over the set (5),

$$\int |\partial_{x_i} u^\#|^2\, dx_1 \leq \int |\partial_{x_i} u|^2\, dx_1, \tag{21}$$

$$\int (u^\#)^2\, dx_1 = \int u^2\, dx_1, \tag{22}$$

$$\int (\alpha\chi_D)_\#(u^\#)^2\, dx_1 \leq \int \alpha\chi_D u^2\, dx_1. \tag{23}$$

Here, $f_\#$ is the increasing symmetric rearrangement of a function f, which is defined by $f_\# = -(-f)^\#$. Note that (21) for $i = 1$ is just the standard rearrangement inequality in one dimension, while for $i > 1$ it is proved as follows: Replace the partial derivatives by difference quotients $(v_\epsilon(x_1) - v_0(x_1))/\epsilon$ with $v_\epsilon(x_1) = u(x_1, \ldots, x_i + \epsilon, \ldots)$. After multplication by ϵ^2 the claimed inequality becomes simply $\int |v_\epsilon^\# - v_0^\#|^2 dx_1 \leq \int |v_\epsilon - v_0|^2 dx_1$ which is well-known.

Fix α and A and assume (u, D) is an optimal pair. Define the set $D^\#$ by $\chi_{D^\#} = (\chi_D)_\#$. Integrating (21), (22) and (23) over x' and summing (21) over i we get

$$\begin{aligned}\lambda(\alpha, D^\#) &\leq \frac{\int_\Omega |\nabla u^\#|^2\, dx + \int_\Omega (\alpha\chi_D)_\# (u^\#)^2\, dx}{\int_\Omega (u^\#)^2\, dx}\\ &\leq \frac{\int_\Omega |\nabla u|^2\, dx + \int_\Omega \alpha\chi_D u^2\, dx}{\int_\Omega u^2\, dx} = \lambda(\alpha; D). \end{aligned} \tag{24}$$

Since we have $|D^\#| = |D| = A$ (by (22) applied to χ_D), optimality of (u, D) implies that $(u^\#, D^\#)$ is also a minimizer and that equality holds in (21) and (23), for all i and almost all x'. We need to show that this implies $u = u^\#$. The statements about D then follow from the characterization $D = \{u \leq t\}$.

First note that since $(u^\#, D^\#)$ is a minimizer, the function $u^\#$ solves the equation $-\Delta u^\# + \alpha\chi_{D^\#} u^\# = \lambda(\alpha; D^\#) u^\#$. Therefore, u and $u^\#$ are continuously differentiable by Theorem 1, so equality in (21) holds for all x'. By a result of Brothers and Ziemer (see [BZ]) this equality implies $u^\#(x_1, x') = u(x_1, x')$ for all x_1 provided the set $\{x_1 : \partial_{x_1} u^\#(x_1, x') = 0\}$ has measure zero.

Therefore, we will be done once we have shown that the set

$$\{v = 0\} \quad \text{has measure zero, where } v = \partial_{x_1} u^\#. \tag{*}$$

We will give two proofs of this: The first proof works whenever $\alpha \neq \overline{\alpha}_\Omega(A)$ and the second proof works whenever $\alpha \leq \overline{\alpha}_\Omega(A)$, so together they cover all cases.

First proof of (*), assuming $\alpha \neq \overline{\alpha}_\Omega(A)$: Assume this was not so. Define $t^\#$ by $D^\# = \{u^\# \leq t^\#\}$. v satisfies $-\Delta v + \alpha\chi_{D^\#} v = \lambda(\alpha, D^\#) v$ on $\{u^\# \neq t^\#\}$. Since $\{u^\# = t^\#\}$ has measure zero by Theorem 1 and the assumption $\alpha \neq \overline{\alpha}_\Omega(A)$, v vanishes on a set of positive measure in the open set $\{u^\# \neq t^\#\}$, so the unique continuation theorem (for sets of positive measure, see [FG]) applied to v implies that $v \equiv 0$ on some connected component K of $\{u^\# \neq t^\#\}$. Therefore, $u^\#$ is constant in the x_1-direction on K. Since $u^\# = 0$ or $t^\#$ on ∂K we conclude that then $u^\#$ must actually be constant on K. This is a contradiction to Theorem 1(c).

Second proof of (*), assuming $\alpha \leq \overline{\alpha}_\Omega(A)$ (this proof is taken from Cox [C]): We show that actually $v < 0$ for $x_1 > 0$, so that $\{v = 0\}$ is contained in the hyperplane $\{x_1 = 0\}$. We have $-\Delta u^\# = \Lambda(\alpha, A) u^\# - \alpha\chi_{D^\#} u^\#$, and the right-hand side is decreasing in x_1 (for $x_1 > 0$) by definition of the rearrangement and since $\alpha \leq \Lambda(\alpha, A)$ by Prop. 10. Taking the x_1-derivative (in the sense of distributions), we get $\Delta v \geq 0$ as distribution. Also, v is continuous, so by the classical theory of subharmonic functions it satisfies the maximum principle (alternatively, it is in H^1 and then the maximum principle as in [GT], Ch. 8, applies). Since $v \leq 0$, we conclude that $v < 0$ unless v vanishes identically in $x_1 > 0$, which is clearly impossible. This proves (*).

This concludes the proof that $u = u^\#$ and hence the proof of the theorem. Note that in the case $\alpha \leq \overline{\alpha}_\Omega(A)$ the second proof of (*) above actually shows that $u_{x_1} < 0$ for $x_1 > 0$. $\square$

Proof of Corollary 5. The only set $D \subset \{|x| < 1\}$ which has the symmetry and convexity properties stated in Theorem 4 in all directions is a shell region as stated. Clearly, $r(A)$ is uniquely determined by A. Therefore, D is unique. □

3.2. Symmetry breaking on annuli. We now give the proof of Theorem 6 about symmetry breaking on an annulus,

$$\Omega = \Omega_a = \{x \in \mathbf{R}^2; a < |x| < a + 1\}, \quad a > 0.$$

Let D be any radial set in Ω,

$$D = \{(r, \theta); r \in D_1, 0 \le \theta < 2\pi\}, \quad D_1 \subset (a, a + 1),$$

and let u be the first eigenfunction for D, with eigenvalue σ:

$$-\Delta u + \alpha \chi_D u = \sigma u \quad \text{on } \Omega, \quad u|_{\partial\Omega} = 0. \tag{25}$$

For a sufficiently large (depending on α and $\delta = |D|/|\Omega|$) we will construct a comparison domain $\tilde{D}$ and a function $\tilde{u}$ which satisfy

$$\frac{\int_{\Omega_a} |\nabla \tilde{u}|^2 + \int_{\Omega_a} \chi_{\tilde{D}} \tilde{u}^2}{\int_{\Omega_a} \tilde{u}^2} \overset{!}{<} \sigma. \tag{26}$$

This shows that D is not an optimal configuration and hence implies the theorem.

In order to construct $\tilde{D}$ and $\tilde{u}$, first pick $N = N(\delta)$ with

$$\delta < 1 - \frac{1}{2N}$$

and consider the sector

$$E_+ = \Omega_a \cap \{(r, \theta); 0 \le \theta \le \pi/N\}.$$

Then let $\tilde{u}$ be the first Dirichlet eigenfunction of the Laplacian on E_+ and $\lambda_1(E_+)$ be the first eigenvalue,

$$-\Delta \tilde{u} = \lambda_1(E_+)\tilde{u} \quad \text{on } E_+, \quad \tilde{u}|_{\partial E_+} = 0, \tag{27}$$

extended by zero on $\Omega \setminus E_+$; the set $\tilde{D}$ can be taken to be any subset of $\Omega \setminus E^+$ with $|\tilde{D}| = |D|$. This is possible since $|D|/|\Omega| = \delta < 1 - \frac{1}{2N} = |\Omega \setminus E_+|/|\Omega|$.

Note that since $\operatorname{supp} \tilde{u} \cap \tilde{D} = \emptyset$, we have $(\int_{\Omega_a} |\nabla \tilde{u}|^2 + \int_{\Omega_a} \chi_{\tilde{D}} \tilde{u}^2)/\int_{\Omega_a} \tilde{u}^2 = \int_{E_+} |\nabla \tilde{u}|^2 / \int_{E_+} \tilde{u}^2 = \lambda_1(E_+)$, so (26) is equivalent to

$$\lambda_1(E_+) \overset{!}{<} \sigma. \tag{28}$$

In order to prove this, we need to introduce a third eigenvalue problem, which is intermediate between (25) and (27).

Define v to be the lowest eigenfunction for the problem (25) among functions of the form

$$v(r, \theta) = h(r) \sin N\theta,$$

and let τ be the associated eigenvalue. Note that problem (25) for such functions is equivalent to the problem

$$-h''(r) - \frac{1}{r}h'(r) + \frac{N^2}{r^2}h(r) + \alpha\chi_{D_1}(r)h(r) = \tau h(r) \quad \text{on } [a, a+1], \tag{29}$$

$$h(a) = h(a+1) = 0 \tag{30}$$

for h. Thus, h is the first eigenfunction of this Sturm-Liouville problem, and the eigenvalue τ is characterized by

$$\tau = \inf_{g \in \mathcal{S}} \frac{\int_a^{a+1}((g')^2 + (\alpha\chi_{D_1} + \frac{N^2}{r^2})g^2)r\,dr}{\int_a^{a+1} g^2 r\,dr}, \tag{31}$$

where $\mathcal{S} = \{g \in C^1[a, a+1];\, g(a) = g(a+1) = 0\}$. From this the (well-known) fact that h does not change sign on $[a, a+1]$ is evident; so we may assume

$$h \geq 0.$$

We will compare u with v and v with $\tilde{u}$. The following two lemmas provide the needed estimates.

Lemma 14. *Let σ be the lowest eigenvalue for the problem* (25) *(with D radial) on $\Omega_{a,b} = \{x \in \mathbb{R}^2 : a < |x| < b\}$, and let τ be the lowest eigenvalue for eigenfunctions of the form $v(r,\theta) = h(r)\sin N\theta$ on $\Omega_{a,b}$. Then we have*

$$\tau - \sigma \leq N^2/a^2.$$

Proof. Since χ_D is assumed radial, the first eigenfunction of (25) is a radial function $u = f(r)$. Now consider the trial function $w(r,\theta) = f(r)\sin N\theta$. We have

$$\tau \leq \frac{\int_{\Omega_{a,b}}(|\nabla w|^2 + \alpha\chi_D w^2)\,dx}{\int_{\Omega_{a,b}} w^2\,dx}.$$

Thus,

$$\tau \leq \frac{\int_a^b((f'(r))^2 + \frac{N^2}{r^2}f(r)^2 + \alpha\chi_{D_1}f(r)^2)r\,dr}{\int_a^b f(r)^2 r\,dr}.$$

By definition of $f(r)$ we get

$$\tau \leq \sigma + \frac{\int_a^b(\frac{N^2}{r^2}f(r)^2)r\,dr}{\int_a^b f(r)^2 r\,dr} \leq \sigma + N^2/a^2.$$

The claim follows. □

Lemma 15. *Define v as above. Assume D is radial and $|D|/|\Omega| = \delta$. There exists a positive constant $c_{\alpha,\delta}$, independent of a, such that for all $a \geq 1$ we have*

$$\frac{\int_D v^2\,dx}{\int_\Omega v^2\,dx} \geq c_{\alpha,\delta}.$$

Proof. We see from $v(r,\theta) = h(r)\sin N\theta$ that

$$\frac{\int_D v^2\,dx}{\int_\Omega v^2\,dx} = \frac{\int_a^{a+1} \chi_{D_1}(r)h(r)^2 r\,dr}{\int_a^{a+1} h(r)^2 r\,dr}. \tag{32}$$

h satisfies Eq. (29). For τ one has a uniform bound $\tau \le C_{\alpha,\delta}$ with $C_{\alpha,\delta}$ independent of $a \ge 1$, because from (31) one gets

$$\tau \le \inf_{g\in\mathcal{S}} \frac{\int_a^{a+1}(g')^2 r\,dr}{\int_a^{a+1} g^2 r\,dr} + \alpha + N^2,$$

and by using for g the translate of any fixed test function on [0, 1] one sees that the first term on the right is bounded by some absolute constant.

Therefore, the coefficients of Eq. (29) are uniformly bounded for $a \ge 1$. Also, we have $h \ge 0$. Lemma 16 in Sect. 6 then implies that one has

$$\inf_{[a+\delta/4,a+1-\delta/4]} h \ge c_{\alpha,\delta}\|h\|_{L^2(a,a+1)}. \tag{33}$$

Since $|D_1| = \delta$, we have $|[a+\delta/4, a+1-\delta/4] \cap D_1| \ge \delta/2$. Therefore,

$$\int_a^{a+1} \chi_{D_1}(r)h(r)^2 r\,dr \ge \frac{\delta}{2}a \inf_{[a+\delta/4,a+1-\delta/4]} h^2 \tag{34}$$

and

$$\int_a^{a+1} h(r)^2 r\,dr \le (a+1)\int_a^{a+1} h^2 \le 2a\int_a^{a+1} h^2. \tag{35}$$

Combining (33), (34) and (35) with (32) we get the lemma. □

End of proof of Theorem 6. We have

$$\tau = \frac{\int_\Omega |\nabla v|^2\,dx}{\int_\Omega v^2\,dx} + \frac{\alpha\int_\Omega \chi_D v^2\,dx}{\int_\Omega v^2\,dx}. \tag{36}$$

Since $v(r,\theta) = h(r)\sin N\theta$, v vanishes on the rays $\theta = 0$ and $\theta = \pi/N$. Since $|v|$ and $|\nabla v|$ are periodic in θ of period π/N, we can replace Ω by E_+ in the first quotient. Therefore, we can use v as test function in the Rayleigh quotient for the Dirichlet Laplacian on E_+ and obtain

$$\frac{\int_\Omega |\nabla v|^2\,dx}{\int_\Omega v^2\,dx} = \frac{\int_{E_+} |\nabla v|^2\,dx}{\int_{E_+} v^2\,dx} \ge \lambda_1(E_+).$$

Combining this with (36) and Lemma 15 we therefore get

$$\tau \ge \lambda_1(E_+) + \alpha c_{\alpha,\delta}. \tag{37}$$

From Lemma 14 we then get

$$\sigma > \tau - N^2/a^2 \ge \lambda_1(E_+) + \alpha c_{\alpha,\delta} - N^2/a^2.$$

If a is chosen so large that $N^2/a^2 \le \alpha c_{\alpha,\delta}$ then this gives (28) and hence the theorem. □

3.3. Symmetry breaking on dumbbells. Proof of Theorem 7. Since α is fixed throughout, we will write $\lambda_\Omega(D) = \lambda_\Omega(\alpha, D)$, $\Lambda_\Omega(A) = \Lambda_\Omega(\alpha, A) = \inf_{|D|=A} \lambda_\Omega(D)$. Here we keep the index Ω since we will also consider these quantities with Ω replaced by one of the 'lobes' $B_\pm = B_1(\pm 2, 0)$. All (implied) constants will only depend on α and A. Write $\Lambda_B = \Lambda_{B_\pm}$, and given D, let

$$D_\pm = D \cap B_\pm, \quad A_\pm = |D_\pm|.$$

Further, we introduce

$$A_{\min} = \min\{\min(|D_-|, |D_+|) : D \subset \Omega, |D| = A\}.$$

Thus, if D is distributed over Ω with the greatest possible imbalance between D_+ and D_- then the smaller of $D_\pm$ will have area $A_{\min}$. It is easily checked that

$$A_{\min} = \max(0, A - |B_-^c|).$$

We first sketch the idea of the proof:

1. For $h = 0$, i.e. two disconnected balls, one clearly has

$$\Lambda_\Omega(A) = \min(\Lambda_B(A_-), \Lambda_B(A_+)). \tag{38}$$

Since Λ_B is strictly increasing, it is optimal to put as much of D as possible in one ball, say B_+, and the "small" remainder in the other. Thus

$$\Lambda_\Omega(A) = \Lambda_B(A_{\min}),$$

and the eigenfunction is zero in B_+.

2. For small positive h, this situation should be approximately the same: Eq. (38) will hold with an error that is a power of h (compare Eq. (42) below), so the same argument as in 1. implies symmetry breaking. Also, the eigenfunction must be small on one lobe, and since $D = \{u \leq t\}$, one gets (b) from an estimate of t.

We now carry out the details. Let (u, D) be an optimal pair. Assume

$$\int_\Omega u^2 = 1.$$

First we need an estimate ensuring that the perturbation introduced by the handle is small. This is provided by the following estimate near the boundary (see [GT, Theorem 8.27 with $R_0 = 1$ and $R = 3h$]), which is applicable since Ω satisfies a uniform exterior cone condition (uniformly in h): There is $\beta \in (0, 1]$ such that

$$\max_{x:\operatorname{dist}(x,\partial\Omega)\leq 3h} u(x) \leq Ch^\beta \|u\|_{L^2(\Omega)}. \tag{39}$$

From this it follows that there is a cut-off function $\sigma = \sigma_h$ on Ω having the following properties:

1. $0 \leq \sigma \leq 1$ on Ω.
2. $\operatorname{supp} \sigma \subset B_- \cup B_+$.
3. $|u| = O(h^\beta)$ on $\operatorname{supp}(1 - \sigma)$.
4. $\int_\Omega |\nabla\sigma|^2 < C$, uniformly as $h \to 0$.

To construct σ, choose $\chi \in C_0^\infty([0,2))$, $0 \le \chi \le 1$, that equals one on $[0, 3/2]$ and set $\sigma(x) = 1 - \chi(|x - (\pm 1, 0)|/h)$ on $B_\pm$ and $\sigma \equiv 0$ on the handle. Properties 1, 2 and 4 are easily checked directly, and property 3 follows from (39).

For brevity, denote, for $\Omega' \subset \Omega$,

$$Q_{\Omega'}(u) = \|\nabla u\|^2_{L^2(\Omega')} + \alpha \|u\|^2_{L^2(D\cap\Omega')}$$

so that $\Lambda_\Omega(A) = Q_\Omega(u)$. Without loss of generality we may assume

$$\Lambda_B(A_-) \le \Lambda_B(A_+).$$

First, we show

$$\Lambda_B(A_{\min}) \ge \Lambda_\Omega(A). \tag{40}$$

This is easy: Take an optimal pair $(\tilde{u}, \tilde{D})$ for $\Lambda_B(A_{\min})$, extend $\tilde{u}$ by zero outside B_- and define a domain $\bar{D} = \tilde{D} \cup D' \subset \Omega$, $|\bar{D}| = A$, by choosing any $D' \subset B_-^c$ with $|D'| = A - A_{\min}$. Since $\tilde{u} \equiv 0$ on D', one gets (40) by using $(\tilde{u}, \bar{D})$ as a test pair for $\Lambda_\Omega(A)$.

Next, we show a reverse inequality. Using the properties of σ and $\operatorname{supp} \nabla\sigma \subset \operatorname{supp}(1-\sigma)$ we obtain, with $\|\cdot\|$ denoting the L^2-norm on $B_\pm$, $\|\nabla(\sigma u)\|^2 = \|\sigma\nabla u + (\nabla\sigma)u\|^2 \le (\|\nabla u\| + \|\nabla\sigma\| \max_{\operatorname{supp}(1-\sigma)} u)^2 \le \|\nabla u\|^2 + O(h^\beta)$ and therefore $Q_{B_\pm}(\sigma u) \le Q_{B_\pm}(u) + O(h^\beta)$. Now we can use σu as test function for the lowest eigenvalue of $-\Delta + \alpha\chi_{D\cap B_\pm}$ on $B_\pm$, and this gives the third inequality in

$$\begin{aligned}
\Lambda_\Omega(A) = Q_\Omega(u) &\ge \sum_\pm Q_{B_\pm}(u) \ge \sum_\pm Q_{B_\pm}(\sigma u) - O(h^\beta)\\
&\ge \sum_\pm \lambda_{B_\pm}(D_\pm) \int_{B_\pm} (\sigma u)^2 - O(h^\beta)\\
&\ge \sum_\pm \lambda_{B_\pm}(D_\pm) \int_{B_\pm} u^2 - O(h^\beta)\\
&\ge \Lambda_B(A_-) + (\Lambda_B(A_+) - \Lambda_B(A_-)) \int_{B_+} u^2 - O(h^\beta).
\end{aligned} \tag{41}$$

In the last two inequalities we have used property 3 of σ, the optimality of $\Lambda_B(A_\pm)$, and $\int_\Omega u^2 = 1$.

Since we assume $\Lambda_B(A_+) \ge \Lambda_B(A_-)$, this and inequality (40) imply

$$\Lambda_B(A_{\min}) \ge \Lambda_\Omega(A) \ge \Lambda_B(A_-) - O(h^\beta). \tag{42}$$

By strict monotonicity of Λ_B one easily gets from this $A_- \le A_{\min} + o(1) \quad (h \to 0)$.

Next, from $D \subset D_+ \cup D_- \cup H$ and $|H| < 4h$ we have $A < A_+ + A_- + 4h$, so $A_+ - A_- > A - 2A_- - 4h \ge A - 2A_{\min} - o(1)$, and then $A_{\min} = \max(0, A - |B_-^c|) \le \max(0, A - \pi)$ gives

$$A_+ - A_- \ge \min(A, 2\pi - A) - o(1). \tag{43}$$

This shows $A_+ \ne A_-$ for $h < h_0(A, \alpha)$ and therefore proves part (a) the theorem.

Now we prove part (b). From (43) we have $A_+ - A_- > c_0$ for some constant $c_0 > 0$, whenever $h < h_0(A, \alpha)$, so strict monotonicity and continuity of Λ_B imply

$$\Lambda_B(A_+) - \Lambda_B(A_-) > c \tag{44}$$

with $c > 0$ independent of h. Now from (40) and (41), and using $\Lambda_B(A_-) \geq \Lambda_B(A_{\min})$ (since $A_- \geq A_{\min}$) and monotonicity, we conclude $(\Lambda_B(A_+) - \Lambda_B(A_-)) \int_{B_+} u^2 = O(h^\beta)$. This and (44) give $\int_{B_+} u^2 = O(h^\beta)$. Since, by (39), $u_{|\partial B_+} = O(h^\beta)$, this L^2 bound implies a pointwise bound for u on B_+ by (47). Combined with (39), applied on the handle, this gives

$$\sup_{x \notin B_-} u(x) = O(h^{\beta/2}). \tag{45}$$

Finally, we want to deduce from (45) that $D^c \subset B_-$ if $A > \pi$ and h is sufficiently small: Since (u, D) is an optimal pair, we have $D = \{u \leq t\}$ for some $t > 0$. Equation (45) shows that we are done if we can show that $t > c$ for a constant $c > 0$ independent of h.

For $r \in (0, 1)$ let $B_-(r)$ be the closed ball of radius r concentric with B_-. Applying Lemma 16 to $G = B_-$ we see, since $\|u\|_{L^2(B_-)} \geq 1 - O(h^\beta)$ by (45), that

$$\inf_{B_-(r)} u \geq c_r \tag{46}$$

for any $r \in (0, 1)$, with $c_r > 0$ only depending on r, A and α, and this implies $|\{u \geq c_r\}| \geq |B_-(r)|$. Therefore, we can conclude $t > c_r$ as soon as $|B_-(r)| > |\Omega| - A$. Since $|\Omega| \leq 2\pi + 4h$ and $A > \pi$, one can find such an r if $h < h_0$, both r and h_0 only depending on A (and α). This completes the proof of the theorem. □

4. Free Boundary and Convex Domains

Proof of Theorem 9, Part (a). First recall, as a consequence of results by Brascamp-Lieb [BL] and Caffarelli-Spruck [CS], that the first eigenfunction ψ on a convex domain possesses only one point where $\nabla\psi = 0$. This point is necessarily the point where ψ attains its maximum. Now given A, we select t_Ω as in Theorem 3, and we select $\delta_0 < t_\Omega$ such that $t_\Omega + \delta_0 < M$ where $M = \max_\Omega \psi$. With this choice of δ_0 we use Theorem 3 to determine a value α_1 for which $[\Omega]^{t_\Omega - \delta_0} \subset D \subset [\Omega]^{t_\Omega + \delta_0}$ for all $\alpha < \alpha_1$. Then the free boundary $\{u = t\}$ is contained in the closed annulus $A = \{t_\Omega - \delta_0 \leq \psi \leq t_\Omega + \delta_0\}$. We have $\nabla\psi \neq 0$ on A, so $C := \min_A |\nabla\psi|$ is positive. Thus decreasing α_1 to a smaller value $\alpha_0 > 0$, we can use Prop. 11 to conclude that for all $\alpha < \alpha_0$ we have $|\nabla u| > C/2$ on A and hence on the free boundary $\{u = t\}$. Applying Theorem 8 we now get the first part of Theorem 9.

Proof of Theorem 9, part (b). We only sketch the proof. Fix x_0 with $\nabla\psi(x_0) \neq 0$. Choose coordinates in which $\nabla\psi(x_0) = (0, \dots, 0, a)$, $a > 0$, and for x' near x_0' (where $x' = (x_1, \dots, x_{n-1})$) and t near $t_0 = \psi(x_0)$ denote the locally unique solution x_n of the equation $\psi(x', x_n) = t$ by $F_0(x', t)$. For α near zero and x near x_0 one has $\partial u_\alpha/\partial x_n \neq 0$ by Prop. 11, so we may define F_α similarly for u_α instead of ψ.

By a result of Korevaar and Lewis [KL] the level set of ψ through x_0 is strictly convex, in the sense that the matrix $(\frac{\partial^2 F_0}{\partial x_i \partial x_j})_{i,j=1,\dots,n-1}$ is positive definite at (x_0', t_0). Therefore, the result follows if one can show continuity of $\frac{\partial^2 F_\alpha}{\partial x_i \partial x_j}$ in α and (x', t). Now the equation for u gives for F_α a uniformly elliptic, quasi-linear equation (writing $y = (x', t)$)

$$\sum_{i,j=1}^{n} b_{ij}(\nabla F_\alpha) \frac{\partial^2 F_\alpha(y)}{\partial y_i \partial y_j} = \alpha \chi_{G_\alpha}(y_n) y_n - \Lambda(\alpha, A) y_n$$

with b_{ij} real analytic and $G_\alpha = (-\infty, t_\alpha]$, where t_α is such that $|\{u_\alpha \leq t_\alpha\}| = A$. From this it is easy to derive the desired regularity, cf. the proof of Lemma 3 in [CGK]. □

5. Numerical Results

In this section we make a few remarks on our method for the numerical solution of our eigenvalue problem.

We use the finite element method for the discretization of our eigenvalue problem, with conforming P-1 elements. To create the mesh we have utilized the automatically spatial meshing program encoded by Y. Tsukuda (see [TK]). In order to calculate the approximate first eigenvalue and the corresponding eigenfunction, we employ the power method.

Our method to obtain an optimal configuration is based on an algorithm that was introduced in [Pi]. However, we do not insist on D (the sought-for optimal configuration) to be a union of elements. This flexibility allows us to find a good approximation even without remeshing.

We now describe the main procedure. The given data are A and α. We first take any initial domain D_0 satisfying $|D_0| = A$. Next, if we have obtained D_{n-1} $(n = 1, 2, 3, \cdots)$ then we calculate the first eigenvalue λ_{n-1} and the corresponding eigenfunction u_{n-1} for the finite element approximation problem for the operator $-\Delta + \alpha\chi_{D_{n-1}}$. Then we obtain D_n from u_{n-1} by finding a number t_0 such that $|\{u_{n-1} \leq t_0\}| = A$ and setting

$$D_n = \{u_{n-1} \leq t_0\}.$$

The number t_0 is determined by a bisection method, i.e. by setting $\mathrm{down}_0 = 0$, $\mathrm{up}_0 = \max_\Omega u_{n-1}$, $j = 0$ and then iterating Steps 1 and 2 (with $L(t) := |\{u_{n-1} \leq t\}|$)

Step 1: Let $\mathrm{interm}_j := (\mathrm{up}_j + \mathrm{down}_j)/2$ and calculate $L(\mathrm{interm}_j)$.

Step 2: If $L(\mathrm{interm}_j) < A$, then $\mathrm{up}_{j+1} := \mathrm{up}_j$ and $\mathrm{down}_{j+1} := \mathrm{interm}_j$, else if $L(\mathrm{interm}_j) > A$, then $\mathrm{up}_{j+1} := \mathrm{interm}_j$ and $\mathrm{down}_{j+1} := \mathrm{down}_j$. Increase j by one.

The iteration is stopped when $L(\mathrm{interm}_k)$ nearly equals A and up_k and down_k nearly equal interm_k according to the adopted precision of approximation, and then we set $t_0 = \mathrm{interm}_k$.

Having obtained D_n we repeat the procedure above to find u_n, D_{n+1} etc. It is easily seen that $\lambda_n \leq \lambda_{n-1}$. We iterate until $|\lambda_n - \lambda_{n-1}| < \epsilon$, where ϵ is given. In the numerical experiments that we have done, we have taken ϵ between 10^{-7} and 10^{-10}.

By the monotonicity of $\{\lambda_n\}$, the limit $\lim_{n\to\infty} = \lambda_\infty$ exists. However, it is not clear a priori whether $\lambda_\infty = \Lambda_\Omega(\alpha, A)$ or not. In order to avoid the latter case, we have repeated the same procedure with several different initial shapes D_0.

The results of some of the computations that we have done are shown in Figs. 1–3. They illustrate well Theorems 2, 4 6, 7, and 9.

6. Some Open Problems and Conjectures

In this section $D = D_{\alpha,A}$ will always denote an optimal configuration.

Conjecture 1 (Uniqueness and convexity). *If Ω is convex then D is unique, and D^c is convex (at least for $\alpha \le \overline{\alpha}_\Omega(A)$).*

Concerning the restriction on α compare the remark after Theorem 2. We have proved convexity for small α in Theorem 9.

Problem 1 (Regularity of the free boundary). *When is the boundary of an optimal configuration smooth everywhere? In general, how can we control the size of singular sets of the free boundary?*

In the convex case we have proved smoothness for small α in Theorem 9. A similar method should easily yield smoothness of the free boundary for small A and smooth $\partial\Omega$.

Conjecture 2. *In dimension two the free boundary ∂D is smooth outside a finite set.*

We prove some restrictions on the singular set of ∂D in [CGK].

Problem 2 (Topology of D and D^c). *If Ω is simply connected, is D also connected even in the case $\alpha > \overline{\alpha}_\Omega(A)$ (cf. Theorem 2)? If A or $|\Omega| - A$ is small enough (with α fixed), is D^c always connected?*

Compare Prop. 12 for the case of A close to $|\Omega|$. In a dumbbell ψ_Ω has two maxima. But numerical evidence suggests the following conjecture:

Conjecture 3 (One component of D^c for dumbbell). *Let Ω_h be a dumbbell. Then for every $\alpha > 0$ there is $\rho_0(\alpha, h) > 0$ such that D^c consists of one component (near one of the maxima of ψ_Ω) whenever $|\Omega| - A < \rho_0(\alpha, h)$.*

Clearly, one would expect $\rho_0(\alpha, h) \to 0$ as $\alpha \to 0$.

We now turn to questions of symmetry. A very general problem is the following:

Problem 3 (Symmetry and symmetry breaking). *Determine (at least qualitatively) the region in the space of parameters where symmetry breaking occurs.*

For annuli the parameters are α, $\delta = A/|\Omega|$ and the ratio τ of outer and inner radius ("thickness"). For dumbbells the parameters are α, A and the thickness of the handle.

First results on this general problem are given by Theorems 6 and 7. See also Fig. 2. The next three conjectures address other aspects of this problem, i.e. they concern other regions in parameter space. They are motivated by numerical experiments.

Conjecture 4 (Symmetry on dumbbells). *Let Ω_h be a dumbbell. Then for every $\alpha > 0$ there is $\rho_1(\alpha, h) > 0$ such that symmetry breaking occurs if and only if $|\Omega| - A < \rho_1(\alpha, h)$.*

Conjecture 5 (Symmetry on annuli). *For each $\alpha, \delta > 0$ there is $\tau_0(\alpha, \delta)$ such that symmetry breaking occurs for the annulus of thickness τ if and only if $\tau < \tau_0(\alpha, \delta)$.*

Theorem 6 gives one half of this. The other half means that the optimal configuration is rotationally symmetric for "thick" annuli. Some aspects of this conjecture are discussed in [CGK].

More generally, it would be interesting to prove symmetry preservation in *any* situation not covered by Theorem 4 (i.e. in a non-convex situation). In particular, a natural conjecture is:

Conjecture 6 (Symmetry preservation for small α). *For any domain Ω and any A there is $\alpha_0(A, \Omega)$ such that for $\alpha \leq \alpha_0(A, \Omega)$ any optimal configuration D has the same symmetries as Ω.*

Also, the analysis of the transition between the symmetric and asymmetric situations would be interesting, as well as the shape of asymmetric solutions for the annulus.

Problem 4 (Relation between D and the curvature of $\partial\Omega$). *Prove that D is fat near points where $\partial\Omega$ has large positive curvature.*

For example see Fig. 1. For $\alpha = 0$ and A near zero this should be not too hard. See [K1] for the case $\alpha = 0$ under additional geometric assumptions. From this one should obtain the result at least for small α and A by perturbation. In [CGK], Thm. 9, we prove in a model case that D is thin near a portion of the boundary which has large negative curvature.

Problem 5 (Limit $\alpha \to \infty$). *Consider the restricted minimization problem, allowing only such sets D for which D^c is a ball. How does this relate to the limit $\alpha \to \infty$ in our problem? Where does the center of an optimal ball lie?*

This is motivated as follows: Formally, for $\alpha = \infty$ the eigenvalue $\lambda_\Omega(\alpha, D)$ equals the first Dirichlet eigenvalue of D^c. (The convergence to this value as $\alpha \to \infty$ is proved in [HH] and [DKM], for example.) Now by the Faber-Krahn inequality (see [Ch], for example), the first Dirichlet eigenvalue of a domain of prescribed area is minimal if the domain is a ball. So the optimal configuration for α large should be close to a ball, at least when A is close enough to $|\Omega|$ (so that a ball of volume $|\Omega| - A$ fits into Ω).

Problem 6 (Other Elliptic Operators). *Consider the same optimization problem for a magnetic Schrödinger operator $(i\nabla - \alpha\chi_D A(x))^2$ with constant magnetic field or a uniformly elliptic operator of divergence type $-\nabla\{(1 + \alpha\chi_D(x))\nabla\}$.*

We have no results for these operators, even if Ω is a ball.

Appendix: Basic PDE Facts

Here we collect some well-known facts about uniform estimates for solutions of elliptic equations. We will state these for an equation

$$Pu = 0, \quad P = \Delta + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c(x), \quad x \in G,$$

where P has measurable, uniformly bounded coefficients, $u \in C^1(G) \cap C^0(\overline{G})$, and $G \subset \mathbb{R}^n$ is a bounded open set. In the following estimates, saying that the constants depend on P will mean that they depend on $\sup_G(b_1, \ldots, b_n, c)$ and stay bounded when this quantity stays bounded.

First, we have the uniform bound (see [GT, Thm. 8.15 and (8.38)])

$$\sup_G |u| \leq C_{G,P}(\|u\|_{L^2(G)} + \sup_{\partial G} |u|). \tag{47}$$

Second, we have Harnack's inequality: If $u \geq 0$ on G and G' is a compact subset of G then

$$\sup_{G'} u \leq C_{G,G',P} \inf_{G'} u. \tag{48}$$

Combining these two we get the following slightly less standard estimate. For $\epsilon > 0$ let $G_\epsilon = \{x \in G : \operatorname{dist}(x, \partial G) \geq \epsilon\}$.

Lemma 16. *For any $\epsilon > 0$ there is a positive constant $c_{G,P,\epsilon}$ such that for any $u \in C^1(G) \cap C^0(\overline{G})$ that solves $Pu = 0$ and satisfies $u \geq 0$ one has*

$$\inf_{G_\epsilon} u \geq c_{G,P,\epsilon}(\|u\|_{L^2(G)} - \sup_{\partial G} u).$$

Here we set $\inf\limits_{\emptyset} u := \infty$.

Proof. We have

$$\begin{aligned}\int_G u^2 &= \int_{G_\epsilon} u^2 + \int_{G \setminus G_\epsilon} u^2 \leq |G_\epsilon| \sup_{G_\epsilon} u^2 + |G \setminus G_\epsilon| \sup_G u^2 \\ &\leq C_{G,P,\epsilon} \inf_{G_\epsilon} u^2 + |G \setminus G_\epsilon|\, C'_{G,P}(\int_G u^2 + \sup_{\partial G} u^2),\end{aligned}$$

where we used Harnack's inequality and the uniform estimate (47). If ϵ is so small that $|G \setminus G_\epsilon|\, C'_{G,P} < 1/2$ then we can subtract the last two terms, and the claim follows easily. The claim for larger ϵ then follows from the fact that $\inf_{G_{\epsilon'}} u \geq \inf_{G_\epsilon} u$ if $\epsilon' \geq \epsilon$. □

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