

Elementary Differential Equations, Section 2

Prof. Loftin: Practice Test Problems for Test 1

1. Give the general solution for the following ordinary differential equations. Solve for y if possible. Show your work.

(a) $y' = \frac{x}{y(x^2+1)}$.

Solution: This equation is separable. Write

$$\begin{aligned}y dy &= \frac{x}{x^2+1} dx, \\ \int y dy &= \int \frac{x}{x^2+1} dx \\ \frac{1}{2}y^2 &= \int \frac{\frac{1}{2}du}{u} = \frac{1}{2} \ln(u) + C = \frac{1}{2} \ln(x^2+1) + C, \\ y^2 &= \ln(x^2+1) + C \text{ (this is a new } C\text{),} \\ y &= \pm \sqrt{\ln(x^2+1) + C}\end{aligned}$$

(b) $\frac{dy}{dx} + 3y = e^{3x}$.

Solution: This is a first-order linear equation.

$$\begin{aligned}\mu(x) &= \exp\left(\int 3 dx\right) = e^{3x}, \\ \mu(x) \left(\frac{dy}{dx} + 3y\right) &= \mu(x)e^{3x}, \\ \frac{d}{dx}(\mu(x)y) &= e^{6x}, \\ \frac{d}{dx}(e^{3x}y) &= e^{6x}, \\ e^{3x}y &= \int e^{6x} dx = \frac{1}{6}e^{6x} + C, \\ y &= e^{-3x} \left(\frac{1}{6}e^{6x} + C\right) = \frac{1}{6}e^{3x} + Ce^{-3x}.\end{aligned}$$

(c) $y'' + 6y' + 5y = 0$.

Solution: This is a second-order linear homogeneous equations with constant coefficients. So consider the characteristic equation

$r^2 + 6r + 5 = 0$, which has roots $r = -1, -5$, to see that two solutions are e^{-t} and e^{-5t} . Then the general solution is

$$c_1 e^{-t} + c_2 e^{-5t}.$$

(d) $y^3 - \sin x + 3xy^2 \frac{dy}{dx} = 0$.

Solution: Let $M = y^3 - \sin x$ and $N = 3xy^2$. Then compute

$$M_y = 3y^2 = N_x.$$

Therefore, this equation is exact. To solve it, we need to find ψ so that $M = \psi_x$, $N = \psi_y$. So take

$$\begin{aligned} \psi &= \int M dx = \int (y^3 - \sin x) dx = xy^3 + \cos x + h(y), \\ N &= \psi_y, \quad 3xy^2 = 3xy^2 + 0 + h'(y), \\ h'(y) &= 0, \quad h = 0, \quad \psi = xy^3 + \cos x. \end{aligned}$$

The solutions are of the form $\psi = xy^3 + \cos x = C$. Solve for y to find

$$y = \sqrt[3]{\frac{C - \cos x}{x}}.$$

2. Find the largest interval on which the solution to the initial value problem

$$(t-1)y''' + 6y'' + (\sin t)y' + (\ln t)y = 0, \quad y(2) = 4, \quad y'(2) = 0, \quad y''(2) = -1$$

must exist. (Do not try to solve the equation.)

Solution: Put the linear equation in the standard form by dividing to make the coefficient of the highest-order term 1:

$$y''' + \frac{6}{t-1} y'' + \frac{\sin t}{t-1} y' + \frac{\ln t}{t-1} y = 0.$$

Now the three coefficient functions are continuous as long as $t - 1 \neq 0$ and $t > 0$ (since $\ln t$ is undefined for $t \leq 0$). This leaves the intervals $(0, 1)$ and $(1, \infty)$. Since our initial conditions are at $t = 2$, the correct interval for the solution is $(1, \infty)$.

3. Solve the initial value problems.

(a) $y' + \frac{y}{t} = \sin t$, $y(\pi) = 1$. (Recall $\sin \pi = 0$, $\cos \pi = -1$.)

Solution: This equation is first-order linear. So compute

$$\begin{aligned}\mu(t) &= \exp\left(\int \frac{1}{t} dt\right) = e^{\ln t} = t, & (\text{assume } t > 0) \\ t\left(y' + \frac{y}{t}\right) &= t \sin t, \\ ty' + y &= t \sin t, \\ (ty)' &= t \sin t, \\ ty &= \int t \sin t dt & (u = t, du = dt), \\ &= t(-\cos t) - \int(-\cos t) dt & (dv = \sin t, v = -\cos t), \\ &= -t \cos t + \sin t + C, \\ y &= -\cos t + \frac{\sin t}{t} + \frac{C}{t}.\end{aligned}$$

Now the initial condition $y = 1$, $t = \pi$ shows

$$\begin{aligned}1 &= -\cos \pi + \frac{\sin \pi}{\pi} + \frac{C}{\pi}, \\ 1 &= -1 + 0 + \frac{C}{\pi}, \\ 2\pi &= C, \\ y &= -\cos t + \frac{\sin t}{t} + \frac{2\pi}{t}.\end{aligned}$$

(b) $4y'' - 9y = 0$, $y(0) = 1$, $y'(0) = 2$.

Solution: The general solution of this second-order linear homogeneous equation with constant coefficients is given by solving the characteristic equation $4r^2 - 9 = 0$. The roots are $r = \frac{3}{2}, -\frac{3}{2}$. Therefore,

$$y = c_1 e^{\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t}.$$

To find the constants, compute

$$\begin{aligned}y' &= \frac{3}{2}c_1 e^{\frac{3}{2}t} - \frac{3}{2}c_2 e^{-\frac{3}{2}t}, \\ 2 = y'(0) &= \frac{3}{2}c_1 e^{\frac{3}{2} \cdot 0} - \frac{3}{2}c_2 e^{-\frac{3}{2} \cdot 0}\end{aligned}$$

$$\begin{aligned}
&= \frac{3}{2}c_1 - \frac{3}{2}c_2, \\
1 = y(0) &= c_1 e^{\frac{3}{2} \cdot 0} + c_2 e^{-\frac{3}{2} \cdot 0} \\
&= c_1 + c_2.
\end{aligned}$$

So we have the two equations $2 = \frac{3}{2}c_1 - \frac{3}{2}c_2$ and $1 = c_1 + c_2$. To solve this, multiply the first equation by $\frac{2}{3}$ to find

$$\begin{aligned}
\frac{4}{3} &= c_1 - c_2, \\
1 &= c_1 + c_2.
\end{aligned}$$

Add the two equations together to find $\frac{7}{3} = 2c_1$ and $c_1 = \frac{7}{6}$. Plug back in to $1 = c_1 + c_2$ to see $c_2 = 1 - \frac{7}{6} = -\frac{1}{6}$. So the solution is

$$y = \frac{7}{6} e^{\frac{3}{2}t} - \frac{1}{6} e^{-\frac{3}{2}t}.$$

(c) $x dx = y dy, \quad y(0) = -1.$

Solution: This equation is separable. Integrate to find

$$\begin{aligned}
\int x dx &= \int y dy, \\
\frac{1}{2}x^2 &= \frac{1}{2}y^2 + C, \\
x^2 &= y^2 + C. \quad ((\text{not the same } C))
\end{aligned}$$

Plug in the initial conditions to find C : $x = 0, y = -1$, to find $0^2 = (-1)^2 + C, C = -1$. So

$$x^2 = y^2 - 1, \quad y^2 = x^2 + 1, \quad y = \pm\sqrt{x^2 + 1}.$$

Since $y(0) = -1 < 0$, choose the negative square root and

$$y = -\sqrt{x^2 + 1}.$$

4. Consider the autonomous equation $y' = y^3 - 4y$.

- (a) Find all the equilibrium solutions to the equation.

Solution: The equilibrium solutions are the roots of $y^3 - 4y = 0$.
Compute

$$y^3 - 4y = y(y^2 - 4) = y(y - 2)(y + 2) = 0,$$

and so the roots are $y = 0, 2, -2$. These are the equilibrium solutions:

$$y = 0, \quad y = 2, \quad y = -2.$$

- (b) Draw a graph in the ty plane containing representative solutions to the equation. Be sure to include the equilibrium solutions, and indicate whether each is stable or unstable.

Solution: The equilibrium solutions break up the y real line into four intervals

$$(-\infty, -2), \quad (-2, 0), \quad (0, 2), \quad (2, \infty).$$

Pick a point in each interval to test the sign of $y' = y^3 - 4y$. Let $f(y) = y^3 - 4y$:

$$f(-3) = (-3)^3 - 4(-3) = -15 < 0,$$

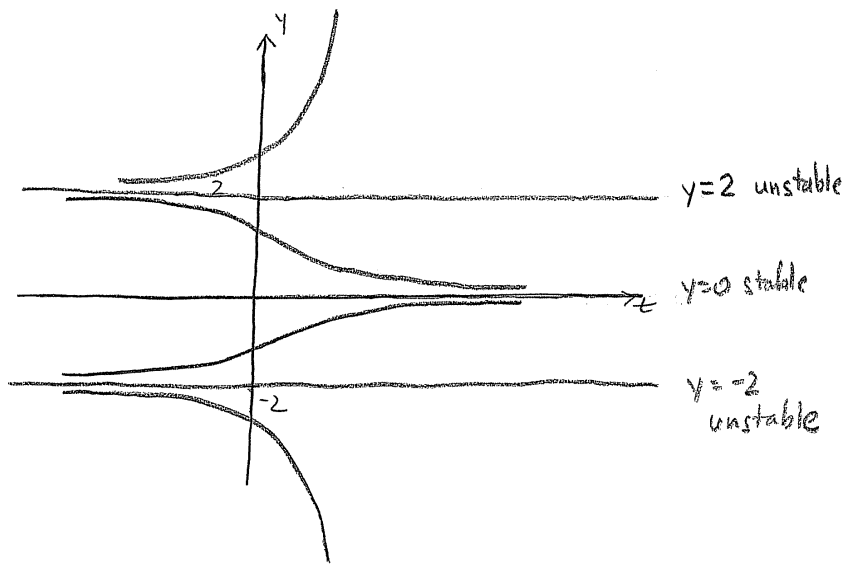
$$f(-1) = (-1)^3 - 4(-1) = 3 > 0,$$

$$f(1) = 1^3 - 4(1) = -3 < 0,$$

$$f(3) = 3^3 - 4(3) = 15 > 0.$$

So $y' = y^3 - 4y < 0$ for y in $(-\infty, -2)$ and $(0, 2)$. Thus any solution y is decreasing there. Similarly, $y' = y^3 - 4y > 0$ for y in $(-2, 0)$ and $(2, \infty)$. So any solution y is increasing there. The picture is as follows:

The equilibrium solution $y = 0$ is stable, while $y = 2$ and $y = -2$ are unstable.



- (c) Compute $\lim_{t \rightarrow \infty} y(t)$ if $y(t)$ is the solution to the initial value problem $y' = y^3 - 4y$, $y(0) = 1$. (You should read this from the graph—do not attempt to solve the equation explicitly.)

Solution: Since the solution y is decreasing for y in $(0, 2)$ (this interval contains the initial value 1), the solution decreases to the lower equilibrium solution $y = 0$ as $t \rightarrow \infty$. In other words,

$$\lim_{t \rightarrow \infty} y(t) = 0.$$

5. Consider the equation $6y + (3x + \frac{2y}{x}) \frac{dy}{dx} = 0$

- (a) Show the equation is not exact.

Solution: Compute $M = 6y$, $N = 3x + \frac{2y}{x}$, and compute $M_y = 6$ and $N_x = 3 - \frac{2y}{x^2}$. Since $M_y \neq N_x$, this equation is not exact.

- (b) Find an integrating factor of the form $\mu(x)$ to make the equation exact.

Solution: We can find an integration factor of the form $\mu(x)$ as long as $\frac{M_y - N_x}{N}$ only depends on x . Compute

$$\frac{M_y - N_x}{N} = \frac{6 - (3 - \frac{2y}{x^2})}{3x + \frac{2y}{x}} = \frac{3 + \frac{2y}{x^2}}{3x + \frac{2y}{x}} = \frac{3 + \frac{2y}{x^2}}{x(3 + \frac{2y}{x^2})} = \frac{1}{x}.$$

Then the integrating factor is given by $\mu' = \mu \frac{1}{x}$, which implies $\mu = x$. So multiplying by $\mu = x$, we find

$$x \left[6y + \left(3x + \frac{2y}{x} \right) \frac{dy}{dx} \right] = 6xy + (3x^2 + 2y) \frac{dy}{dx} = 0.$$

- (c) Use the integrating factor to find the general solution to the equation.

Solution: To check the new equation is exact, note $M = 6xy$, $N = 3x^2 + 2y$ and $M_y = 6x$, $N_x = 6x$. To find ψ so that $\psi_x = M$ and $\psi_y = N$, compute

$$\begin{aligned} \psi &= \int M dx = \int 6xy dx = 3x^2y + h(y), \\ \psi_y &= 3x^2 + h'(y) = N = 3x^2 + 2y, \\ h'(y) &= 2y, \quad h = y^2, \\ \psi &= 3x^2y + y^2. \end{aligned}$$

So the general solution is $3x^2y + y^2 = C$. It's possible to use the quadratic equation to solve for y .

6. A lake holds 10^{10} liters of water, and the lake water is currently polluted with 1 mg of a pollutant per liter. A stream of clean water carrying 10^7 liters per day enters the lake. 2×10^6 liters of water evaporate from the lake each day (the pollution does not evaporate), while an outgoing stream takes 8×10^6 liters of (mixed) lake water away from the lake each day.

How long does it take until the pollution level in the lake drops to an acceptable concentration of .1 mg/liter? Show your work, and use a calculator.

Solution: Let P be the amount of pollutant in the lake, in mg. So

$$P(0) = 10^{10} \text{ liter} \cdot \frac{1 \text{ mg}}{\text{liter}} = 10^{10} \text{ mg}.$$

Note that the amount of water in the lake remains constant: 10^7 liters come in, and $2 \times 10^6 + 8 \times 10^6 = 10^7$ liters go out. The rate in of the pollutants is 0, while the rate out is given by

$$\frac{P \text{ mg}}{10^{10} \text{ liter}} \cdot \frac{8 \times 10^6 \text{ liter}}{\text{day}} = \frac{(8 \times 10^{-4})P \text{ mg}}{\text{day}}.$$

Thus the differential equation is

$$\frac{dP}{dt} = \text{rate in} - \text{rate out} = 0 - (8 \times 10^{-4})P = -.0008P,$$

with the initial condition $P(0) = 10^{10}$. The solution to this initial value problem is

$$P(t) = P(0)e^{-.0008t} = 10^{10}e^{-.0008t}.$$

We are asked to find the time t when the concentration falls to .1 mg/liter, which corresponds to $.1(10^{10}) = 10^9$ mg. This corresponds to

$$\begin{aligned} P(t) &= 10^9, \\ 10^{10}e^{-.0008t} &= 10^9, \\ e^{-.0008t} &= .1, \\ t &= \frac{\ln .1}{-.0008} \sim 2878 \text{ days}, \end{aligned}$$

which is about 7.88 years.