Elementary Differential Equations, Section 2
Prof. Loftin: Practice Test Problems for Test 2

SOLUTIONS

1. Find the radius of convergence of the power series

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots+\frac{x^n}{n}+\cdots$$

Show your work.

Solution: The general term is $b_n=\frac{x^n}{n}$. Compute

$$R=\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|=\lim_{n\to\infty}\left|\frac{\left(\frac{x^{n+1}}{n+1}\right)}{\left(\frac{x^n}{n}\right)}\right|=\lim_{n\to\infty}\frac{|x|n}{n+1}=|x|\lim_{n\to\infty}\frac{n}{n+1}=|x|.$$

So the Ratio Test says the series converges if $R=|x|<1$. So the radius of convergence is 1.

2. (a) Find the general solution to $y''+4y=0$.

Solution: If $y=e^{rt}$, the characteristic equation is $r^2+4=0$. The roots are $r=\pm 2i$. So a fundamental set of solutions is $\{\cos 2t,\sin 2t\}$, and the general solution is

$$c_1\cos 2t+c_2\sin 2t.$$

(b) Use the method of undetermined coefficients to find a solution to $y''+4y=\sin 2t$.

Solution: By part (a), $\sin 2t$ is a solution to the homogeneous equation. Therefore, in order for the method of undetermined coefficients to be successful, we must multiply the standard functions by t. So write

$$\begin{aligned}y &= At\sin 2t+Bt\cos 2t,\\ y' &= A\sin 2t+2At\cos 2t+B\cos 2t-2Bt\sin 2t,\end{aligned}$$

$$\begin{aligned}
y'' &= 2A\cos 2t + 2A\cos 2t - 4At\sin 2t - 2B\sin 2t - 2B\sin 2t - 4Bt\cos 2t \\
&= 4A\cos 2t - 4At\sin 2t - 4B\sin 2t - 4Bt\cos 2t, \\
\sin 2t &= y'' + 4y = (4A\cos 2t - 4At\sin 2t - 4B\sin 2t - 4Bt\cos 2t) \\
&\quad + 4(At\sin 2t + Bt\cos 2t) \\
&= 4A\cos 2t - 4B\sin 2t.
\end{aligned}$$

Therefore, comparing coefficients, $4A = 0$ for the $\cos 2t$ coefficients, and $-4B = 1$ for the $\sin 2t$ coefficients. So $A = 0$, $B = -\frac{1}{4}$, and

$$y = -\frac{1}{4}t\cos 2t.$$

(c) Find the general solution to $y'' + 4y = \sin 2t$.

Solution: The general solution to the nonhomogeneous equations is the particular solution plus the general solution to the homogeneous problem:

$$y = -\frac{1}{4}t\cos 2t + c_1\cos 2t + c_2\sin 2t.$$

(d) Find the solution to the initial value problem

$$y'' + 4y = \sin 2t, \qquad y(0) = 2, \qquad y'(0) = 1.$$

Solution: If y is the general solution from part (c), compute

$$\begin{aligned}
2 &= y(0) = -\frac{1}{4}(0)\cos(2\cdot 0) + c_1\cos(2\cdot 0) + c_2\sin(2\cdot 0) = c_1, \\
y' &= -\frac{1}{4}\cos 2t + \frac{1}{2}t\sin 2t - 2c_1\sin 2t + 2c_2\cos 2t, \\
1 &= y'(0) = -\frac{1}{4}\cos(2\cdot 0) + \frac{1}{2}(0)\sin(2\cdot 0) - 2c_1\sin(2\cdot 0) + 2c_2\cos(2\cdot 0) \\
&= -\frac{1}{4} + 2c_2, \\
c_2 &= \frac{5}{8}, \\
y &= -\frac{1}{4}t\cos 2t + 2\cos 2t + \frac{5}{8}\sin 2t.
\end{aligned}$$

3. Find the general solution to $y''' - y'' + y' - y = 0$. Show your work.

Solution: If $y = e^{rt}$, the characteristic equation is

$$r^3 - r^2 + r - 1.$$

To factor this, we can check the guesses for roots ± 1 to find that $r = 1$ is a root. Dividing, we find $r^3 - r^2 + r - 1 = (r-1)(r^2+1)$, and so the roots are $r = 1$, $r = \pm i$. So a fundamental set of solutions is

$$\{e^t, \cos t, \sin t\}$$

and the general solution is

$$c_1 e^t + c_2 \cos t + c_3 \sin t.$$

4. Find the recurrence relation for a power series around $x = 0$ for the equation $y'' - xy' + y = 0$. Find the general term of the power series. Show your work.

Solution: Compute

$$\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\
y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots \\
y'' &= 2a_2 + 3(2)a_3 x + \cdots + n(n-1)a_n x^{n-2} + \cdots \\
-xy' &= -a_1 x - 2a_2 x^2 - 3a_3 x^3 - \cdots - na_n x^n - \cdots
\end{aligned}$$

We need to adjust the general term of y'' by substituting $n \to (n+2)$ to be $(n+2)(n+1)a_{n+2}x^n$. All together, we have

$$\begin{array}{cccccccccc}
0 = & & & & & & & & & \\
 & 2a_2 & + & 3(2)a_3 x & + & 4(3)a_4 x^2 & + \cdots + & (n+2)(n+1)a_{n+2}x^n & + \cdots \\
 & & - & a_1 x & - & 2a_2 x^2 & - \cdots - & na_n x^n & - \cdots \\
+ & a_0 & + & a_1 x & + & a_2 x^2 & + \cdots + & a_n x^n & + \cdots
\end{array}$$

So the recurrence relation, given by the coefficients of the x^n terms, is

$$(n+2)(n+1)a_{n+2} - na_n + a_n = 0, \qquad a_{n+2} = \frac{n-1}{(n+2)(n+1)}\, a_n.$$

To find the general term, a_0 and a_1 will be free, and we can use the equations above to get started. Constant terms give us

$$2a_2 + a_0 = 0, \qquad a_2 = -\frac{1}{2}a_0.$$

x terms give

$$3(2)a_3 - a_1 + a_1 = 0, \qquad a_3 = 0.$$

x^2 terms give

$$4(3)a_4 - 2a_2 + a_2 = 0, \qquad a_4 = \frac{a_2}{4(3)} = -\frac{1}{4(3)(2)}\,a_0.$$

We can use the recurrence relation to go farther:

$$a_5 = \frac{2}{5 \cdot 4}a_3 = 0,$$

and all odd coefficients beyond a_1 are 0. Also

$$a_6 = \frac{3}{6 \cdot 5}a_4 = -\frac{3}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}\,a_0 = -\frac{3}{6!}\,a_0.$$

The even terms in general are given by

$$a_{2n} = \frac{(-1)(1)\cdots(2n-3)}{(2n)!}\,a_0.$$

All together,

$$y = a_1 x + a_0\left(1 - \frac{1}{2}\,x^2 - \frac{1}{4!}\,x^4 - \frac{3}{6!}\,x^6 + \cdots + \frac{(-1)(1)\cdots(2n-3)}{(2n)!}\,x^{2n} + \cdots\right)$$

5. Find the general solution to $x^2y'' + 7xy' + 5y = 0$. Show your work.

Solution: If $y = x^r$, we find

$$0 = r(r-1) + 7r + 5 = r^2 + 6r + 5 = (r+5)(r+1),$$

and so a fundamental set of solutions is $|x|^{-5}$ and $|x|^{-1}$. The general solution is

$$c_1|x|^{-5} + c_2|x|^{-1}.$$

6. Find all the singular points of $(x^4 - x^2)y'' + (x+2)y' + y = 0$. Classify each as a regular or irregular singular point, and show your work.

Solution: Normalize

$$y'' + \frac{x+2}{x^4 - x^2}\, y' + \frac{1}{x^4 - x^2}\, y = y'' + \frac{x+2}{x^2(x-1)(x-2)}\, y' + \frac{1}{x^2(x-1)(x+1)}\, y = 0.$$

So the singular points are at the roots $x = 0$, $x = 1$, $x = -1$.

The singularity at $x = 1$ is like $\frac{1}{x-1}$ for both the y' and y coefficients. This is allowed for a regular singular point (in fact $\frac{1}{(x-1)^2}$ would be allowed for the y coefficient). So $x = 1$ is a regular singular point.

$x = -1$ works the same way, and so it is a regular singular point.

For $x = 0$, however, the singularity of the y' coefficient is like $\frac{1}{x^2}$, which is not allowed for a regular singular point. Thus $x = 0$ is an irregular singular point of the equation (even though the y coefficient is like $\frac{1}{x^2}$, which is OK).

7. Use Abel's Theorem to determine the Wronskian of two solutions to the differential equation $xy'' + y' + 3y = 0$ (for $x > 0$). Show your work.

Solution: Upon normalizing the equation to be

$$y'' + \frac{1}{x}\, y' + \frac{3}{x}\, y = y'' + p(x)y' + q(x)y = 0,$$

Abel's Theorem states that the Wronskian of any two solutions is equal to

$$c \exp\left(\int -p(x)\, dx\right) = c \exp\left(\int -\frac{1}{x}\, dx\right) = c \exp(-\ln x) = cx^{-1}$$

for a constant c.

8. Find all the terms up to the a_4 term of power series representation of the solution to

$$y'' + 6xy' + 4y = x, \qquad y(0) = 1, \qquad y'(0) = 3.$$

Show your work.

Solution: Compute with the standard power series

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ y' &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\ y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \end{aligned}$$

Then we plug in the initial conditions to find $1 = y(0) = a_0$, $3 = y'(0) = a_1$. Also, the equation $y'' + 6xy' + 4y = x$ becomes

$$\begin{array}{cccccccc} & 2a_2 & + & 6a_3 x & + & 12a_4 x^2 & + \cdots \\ & & + & 6a_1 x & + & 12a_2 x^2 & + \cdots \\ + & 4a_0 & + & 4a_1 x & + & 4a_2 x^2 & + \cdots \\ = & & & x & & & \end{array}$$

The constant terms then give

$$2a_2 + 4a_0 = 0, \qquad a_2 = -2a_0 = -2.$$

The x terms imply

$$6a_3 + 6a_1 + 4a_1 = 1, \qquad a_3 = \frac{1}{6}(-10a_1 + 1) = \frac{1}{6}(-10(3) + 1) = -\frac{29}{6}.$$

Finally, the x^2 terms imply

$$12a_4 + 12a_2 + 4a_2 = 0, \qquad a_4 = \frac{1}{12}(-16a_2) = \frac{1}{12}(-16)(-2) = \frac{8}{3}.$$

So the solution

$$y \sim 1 + 3x - 2x^2 - \frac{29}{6}x^3 + \frac{8}{3}x^4.$$