Elementary Differential Equations, Section 2
Prof. Loftin: Practice Test Problems for the Material after Test 2

SOLUTIONS

1. Consider the following function

$$f(t) = \begin{cases} 1 & \text{for } \quad t < 2 \\ t+1 & \text{for } 2 \le t < 5 \\ 3 & \text{for } \quad t \ge 5 \end{cases}$$

   (a) Rewrite $f(t)$ using a single formula in terms of $u_2(t)$ and $u_5(t)$.
   Solution: Compute

$$\begin{aligned} f(t) &= 1[1 - u_2(t)] + (t+1)[u_5(t) - u_2(t)] + 3u_5(t) \\ &= 1 - tu_2(t) + (t-2)u_5(t) \end{aligned}$$

   (b) Find the Laplace transform of $f(t)$.
   Solution: Compute

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{1\} - \mathcal{L}\{tu_2(t)\} + \mathcal{L}\{(t-2)u_5(t)\} \\ &= \frac{1}{s} - \mathcal{L}\{[(t-2)+2]u_2(t)\} + \mathcal{L}\{[(t-5)+3]u_5(t)\} \\ &= \frac{1}{s} - e^{-2s}\mathcal{L}\{t+2\} + e^{-5s}\mathcal{L}\{t+3\} \\ &= \frac{1}{s} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-5s}\left(\frac{1}{s^2} + \frac{3}{s}\right). \end{aligned}$$

   (c) Use the Laplace transform to find the solution to the initial value problem

$$y' - 2y = f(t), \qquad y(0) = 1.$$

   Show your work.
   Solution: Compute for $\mathcal{L}\{y\} = F(s)$

$$\mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{f(t)\},$$

$$
\begin{aligned}
sF(s) - y(0) - 2F(s) &= \frac{1}{s} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-5s}\left(\frac{1}{s^2} + \frac{3}{s}\right), \\
(s-2)F(s) &= 1 + \frac{1}{s} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-5s}\left(\frac{1}{s^2} + \frac{3}{s}\right), \\
F(s) &= \frac{1}{s-2} + \frac{1}{s(s-2)} - e^{-2s}\left(\frac{1}{s^2(s-2)} + \frac{2}{s(s-2)}\right) \\
&\quad + e^{-5s}\left(\frac{1}{s^2(s-2)} + \frac{3}{s(s-2)}\right).
\end{aligned}
$$

We may proceed by doing partial fractions on $\frac{1}{s(s-2)}$ and $\frac{1}{s^2(s-2)}$:

$$
\begin{aligned}
\frac{1}{s(s-2)} &= \frac{a}{s} + \frac{b}{s-2}, \\
1 &= a(s-2) + bs, \\
1 &= a(2-2) + b(2), \qquad b = \frac{1}{2}, \\
1 &= a(0-2) + b(0), \qquad a = -\frac{1}{2}, \\
\frac{1}{s(s-2)} &= \frac{-\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s-2}, \\
\frac{1}{s^2(s-2)} &= \frac{cs+d}{s^2} + \frac{e}{s-2}, \\
1 &= (cs+d)(s-2) + es^2 = (c+e)s^2 + (-2c+d)s - 2d, \\
1 &= -2d, \qquad d = -\frac{1}{2}, \qquad (1 \text{ terms}) \\
0 &= -2c + d, \qquad c = \frac{1}{2}d = -\frac{1}{4}, \qquad (s \text{ terms}) \\
0 &= c + e, \qquad e = -c = \frac{1}{4}, \qquad (s^2 \text{ terms}) \\
\frac{1}{s^2(s-2)} &= \frac{-\frac{1}{4}s - \frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s-2} = -\frac{\frac{1}{4}}{s} - \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s-2},
\end{aligned}
$$

So we compute

$$
\begin{aligned}
y &= \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\} \\
&\quad - \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s^2(s-2)} + \frac{2}{s(s-2)}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& + \mathcal{L}^{-1}\left\{ e^{-5s}\left(\frac{1}{s^2(s-2)} + \frac{3}{s(s-2)} \right) \right\} \\
&= e^{2t} + \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s-2} \right\} + u_2(t)h(t-2) + u_5(t)k(t-5), \\
h(t) &= \mathcal{L}^{-1}\left\{ \frac{1}{s^2(s-2)} + \frac{2}{s(s-2)} \right\} \\
&= \mathcal{L}^{-1}\left\{ \left[\frac{-\frac{1}{4}}{s} - \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s-2} \right] + 2\left[\frac{-\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s-2} \right] \right\} \\
&= \left[-\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t} \right] + 2\left[-\frac{1}{2} + \frac{1}{2}e^{2t} \right] \\
&= -\frac{5}{4} - \frac{1}{2}t + \frac{5}{4}e^{2t}, \\
k(t) &= \mathcal{L}^{-1}\left\{ \frac{1}{s^2(s-2)} + \frac{3}{s(s-2)} \right\} \\
&= \left[-\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t} \right] + 3\left[-\frac{1}{2} + \frac{1}{2}e^{2t} \right] \\
&= -\frac{7}{4} - \frac{1}{2}t + \frac{7}{4}e^{2t}, \\
y &= e^{2t} - \frac{1}{2}\cdot 1 + \frac{1}{2}e^{2t} + u_2(t)\left(-\frac{5}{4} - \frac{1}{2}(t-2) + \frac{5}{4}e^{2(t-2)} \right) \\
& +u_5(t)\left(-\frac{7}{4} - \frac{1}{2}(t-5) + \frac{7}{4}e^{2(t-5)} \right).
\end{aligned}
$$

2. Determine all the singular points of the following differential equation:

$$x^2(x-1)^2(x+2)y'' + (x+x^3)y' + y = 0.$$

Determine which of the singular points are regular. For each regular singular point, calculate the indicial equation and its roots (i.e., the exponents). Show your work.

Solution: Since the coefficients of y' and y are polynomials, the only singular points are given by the roots of the coefficient of y'', given by $x^2(x-1)^2(x+2) = 0$. The roots are $x = 0, 1, -2$. These are the singular points of the equation.

To see if $x = 0$ is regular, divide to find

$$0 = y'' + \frac{x+x^3}{x^2(x-1)^2(x+2)}y' + \frac{1}{x^2(x-1)^2(x+2)}y$$

$$= \quad y'' + \frac{1+x^2}{x(x-1)^2(x+2)}\,y' + \frac{1}{x^2(x-1)^2(x+2)}\,y$$

The singularity of the y' coefficient is like x^{-1}, and the y coefficient has singularity like x^{-2}. This is allowed for a regular singular point, and $x = 0$ is regular.

Similarly, for $x = 1$, the y' coefficient has a singularity like $(x-1)^{-2}$, and so $x = 1$ is not a regular singular point.

For $x = -2$, both the y' and y singularities are like $(x+2)^{-1}$, which is allowed for a regular singular point. So $x = -2$ is regular.

For the regular singular point $x = 0$, write the equation as

$$\begin{aligned} 0 &= x^2y'' + x\,\frac{1+x^2}{(x-1)^2(x+2)}\,y' + \frac{1}{(x-1)^2(x+2)}\,y \\ &= x^2y'' + x(p_0 + p_1x + \cdots)y' + (q_0 + q_1x + \cdots)y \end{aligned}$$

So

$$p_0 = \left.\frac{1+x^2}{(x-1)^2(x+2)}\right|_{x=0} = \frac{1}{2}, \qquad q_0 = \left.\frac{1}{(x-1)^2(x+2)}\right|_{x=0} = \frac{1}{2}.$$

Thus the indicial equation

$$0 = r(r-1) + \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{1}{2}r + \frac{1}{2}$$

has roots

$$\frac{\frac{1}{2} \pm \sqrt{(\frac{1}{2})^2 - 4\cdot 1\cdot \frac{1}{2}}}{2\cdot 1} = \frac{1}{4} \pm \frac{\sqrt{7}}{4}\,i.$$

For the regular singular point $x = -2$, write the equation as

$$\begin{aligned} 0 &= (x+2)^2y'' + (x+2)\,\frac{1+x^2}{x(x-1)^2}\,y' + \frac{x+2}{x^2(x-1)^2}\,y \\ &= (x+2)^2y'' + (x+2)[p_0 + p_1(x+2) + \cdots]\,y' + [q_0 + q_1(x+2) + \cdots]\,y. \end{aligned}$$

So

$$p_0 = \left.\frac{1+x^2}{x(x-1)^2}\right|_{x=-2} = -\frac{5}{18}, \qquad q_0 = \left.\frac{x+2}{x^2(x-1)^2}\right|_{x=-2} = 0.$$

Thus the indicial equation

$$0 = r(r-1) - \frac{5}{18}r + 0 = r^2 - \frac{23}{18}r$$

The roots are $r = 0, \frac{23}{18}$.

3. Consider the differential equation

$$xy'' + (1+2x)y' + y = 0.$$

(a) Show that $x = 0$ is a regular singular point, and determine the indicial equation. Show that 0 is a double root of the indicial equation.

Solution: Put the equation in the standard form

$$x^2y'' + x(1+2x)y' + xy = 0.$$

So $p_0 + p_1x + \cdots = 1 + 2x$ and so $p_0 = 1$, and $q_0 + q_1x + \cdots = x$ and $q_0 = 0$. (This shows $x = 0$ is a regular singular point, since we can put these coefficients of y' and y in the correct format.) So the indicial equation is

$$0 = r(r-1) + 1 \cdot r + 0 = r^2.$$

So $r = 0$ is a double root.

(b) Find the first three terms of a solution to the equation of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

Solution: Compute

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + \cdots, \\ y' &= a_1 + 2a_2x + \cdots, \\ y'' &= 2a_2 + \cdots, \end{aligned}$$

Write the equation as $xy'' + y' + 2xy' + y = 0$, and so

$$\begin{array}{rcccl} 0 = & & & 2a_2x & +\cdots \\ + & a_1 & + & 2a_2x & +\cdots \\ + & & & 2a_1x & +\cdots \\ + & a_0 & + & a_1x & +\cdots \end{array}$$

The constant terms give $a_0 + a_1 = 0$, and $a_1 = -a_0$. The x terms give $4a_2 + 3a_1 = 0$, and $a_2 = -\frac{3}{4}a_1 = \frac{3}{4}a_0$. So the solution is

$$a_0 - a_0 x + \frac{3}{4}a_0 x^2 + \cdots = a_0\left(1 - x + \frac{3}{4}x^2 + \cdots\right).$$

(c) Find the first three terms of the general solution to the equation, by finding all solutions of the form

$$a y_1(x)\ln x + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots,$$

where $y_1(x)$ is the solution to part (b) with $a_0 = 1$.

Solution: So $y_1(x) = 1 - x + \frac{3}{4}x^2 + \cdots$, and compute

$$\begin{aligned}
y &= a\left(1 - x + \frac{3}{4}x^2 + \cdots\right)\ln x + b_0 + b_1 x + b_2 x^2 + \cdots \\
&= a\ln x + b_0 - ax\ln x + b_1 x + \frac{3}{4}ax^2\ln x + b_2 x^2 + \cdots, \\
y' &= ax^{-1} - a\ln x + (-a + b_1) + \frac{3}{2}ax\ln x + \left(\frac{3}{4}a + 2b_2\right)x + \cdots, \\
y'' &= -ax^{-2} - ax^{-1} + \frac{3}{2}a\ln x + \left(\frac{9}{4}a + 2b_2\right) + \cdots
\end{aligned}$$

To compute via the equation $xy'' + y' + 2xy' + y = 0$ to find

$$\begin{array}{rrrrrrrrrrr}
0 & = & -ax^{-1} & & & - & a & + & \frac{3}{2}ax\ln x & + & \left(\frac{9}{4}a + 2b_2\right)x & + \cdots \\
& + & ax^{-1} & - & a\ln x & + & (-a + b_1) & + & \frac{3}{2}ax\ln x & + & \left(\frac{3}{4}a + 2b_2\right)x & + \cdots \\
& & & & & + & 2a & - & 2ax\ln x & + & (-2a + 2b_1)x & + \cdots \\
& & & + & a\ln x & + & b_0 & - & ax\ln x & + & b_1 x & + \cdots
\end{array}$$

The x^{-1} and $\ln x$ terms do not impose any restrictions, while the constant terms imply

$$-a + (-a + b_1) + 2a + b_0 = 0, \qquad b_1 = -b_0.$$

The $x\ln x$ terms do not impose a restriction, while the x terms give

$$\frac{9}{4}a + 2b_2 + \frac{3}{4}a + 2b_2 + (-2a + 2b_1) + b_1 = 0, \qquad b_2 = -\frac{1}{4}a - \frac{3}{4}b_1 = -\frac{1}{4}a + \frac{3}{4}b_0.$$

So, altogether, we have the general solution as

$$\begin{aligned} y &= ay_1(x)\ln x + b_0 - b_0 x + \left(-\frac{1}{4}a + \frac{3}{4}b_0\right)x^2 + \cdots \\ &= a\left(y_1(x)\ln x - \frac{1}{4}x^2 + \cdots\right) + b_0\left(1 - x + \frac{3}{4}x^2 + \cdots\right) \\ &= a\left(y_1(x)\ln x - \frac{1}{4}x^2 + \cdots\right) + b_0 y_1(x). \end{aligned}$$

4. Derive the formula for the Laplace transform $\mathcal{L}\{t\}$. (In other words, compute it from the definition, and don't just look it up in the table.)

   Solution: Compute

$$\begin{aligned} \mathcal{L}\{t\} &= \int_0^\infty e^{-st}t\,dt \qquad\qquad u = t,\ dv = e^{-st}dt \\ &= \lim_{T\to\infty}\int_0^T e^{-st}t\,dt \qquad du = dt,\ v = -\frac{1}{s}e^{-st}, \\ &= \lim_{T\to\infty}\left(-\frac{1}{s}te^{-st}\Big|_0^T - \int_0^T -\frac{1}{s}e^{-st}dt\right) \\ &= \lim_{T\to\infty}\left(-\frac{1}{s}Te^{-sT} + \frac{1}{s}0\,e^{s\cdot 0} - \frac{1}{s^2}e^{-st}\Big|_0^T\right) \\ &= \lim_{T\to\infty}\left(-\frac{1}{s}Te^{sT} - \frac{1}{s^2}e^{-sT}\right) + 0 + \frac{1}{s^2} \\ &= \frac{1}{s^2} \qquad \text{for } s > 0. \end{aligned}$$

5. Consider the differential equation

$$2x^2y'' + xy' + (x - x^2)y = 0.$$

   (a) Write down the indicial equation for $x = 0$ and find its roots r_1, r_2.

   Solution: The indicial equation is $2r(r-1) + r + 0 = 0$, and the roots are $r_1 = \frac{1}{2}$, $r_2 = 0$.

   (b) Find the recurrence relation for a solution of the differential equation of the form

$$|x|^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots),$$

where r is a root of the indicial equation.

Solution: For $x > 0$, write

$$\begin{aligned} y &= a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots, \\ y' &= r a_0 x^{r-1} + (r+1) a_1 x^r + \cdots + (r+n) a_n x^{r+n-1} + \cdots, \\ y'' &= r(r-1) a_0 x^{r-2} + (r+1) r a_1 x^{r-1} + \cdots + (r+n)(r+n-1) a_n x^{r+n-2} + \cdots, \end{aligned}$$

The equation is $2x^2y'' + xy' + xy - x^2y = 0$, and so we have the following general terms:

$$\begin{aligned} 2x^2y'' &: 2(r+n)(r+n-1)a_n x^{r+n}, \\ xy' &: (r+n)a_n x^{r+n}, \\ xy &: a_n x^{r+n+1}, \\ -x^2y &: -a_n x^{r+n+2}. \end{aligned}$$

We need to shift the last two terms to get the x^{r+n} power: $a_{n-1}x^{r+n}$ and $-a_{n-2}x^{r+n}$. So the recurrence relation is

$$2(r+n)(r+n-1)a_n + (r+n)a_n + a_{n-1} - a_{n-2} = 0,$$

and

$$a_n = \frac{-a_{n-1} + a_{n-2}}{2(r+n)(r+n-1) + (r+n)} = \frac{-a_{n-1} + a_{n-2}}{2(r+n)^2 - (r+n)}.$$

(c) For each exponent r_1, r_2, write down the first three nonzero terms of the solution to the differential equation.

Solution: For $r_1 = \frac{1}{2}$, choose $a_0 = c_1$, and apply the recursion formula. (Note that $a_{-1} = 0$, as there is no such term.)

$$\begin{aligned} a_1 &= \frac{-a_0 + a_{-1}}{2(\frac{3}{2})^2 - (\frac{3}{2})} = -\frac{1}{3}c_1 \\ a_2 &= \frac{-a_1 + a_0}{2(\frac{5}{2})^2 - (\frac{5}{2})} = \frac{2}{15}c_1. \end{aligned}$$

So the solution is

$$|x|^{\frac{1}{2}} c_1 \left(1 - \frac{1}{3}x + \frac{2}{15}x^2 + \cdots\right).$$

For $r_2 = 0$, choose $a_0 = c_2$ and apply the recursion formula (again $a_{-1} = 0$).

$$\begin{aligned} a_1 &= \frac{-a_0 + a_{-1}}{2 \cdot 1^2 - 1} = -c_2, \\ a_2 &= \frac{-a_1 + a_0}{2 \cdot 2^2 - 2} = \frac{1}{3} c_2. \end{aligned}$$

So the solution is

$$c_2 \left(1 - x + \frac{1}{3} x^2 + \cdots \right).$$