

Elementary Differential Equations: Quiz #9, April 14, 2010

SOLUTION

1. For the following equation, show 0 is a regular singular point. Write down the first three nonzero terms of two linearly independent solutions of the form

$$|x|^r(a_0 + a_1x + a_2x^2 + \dots)$$

for each root r of the indicial equation.

$$x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0.$$

Solution: To show 0 is a regular singular point, rewrite

$$y'' + \frac{1}{x}y' + \frac{x^2 - \frac{4}{9}}{x^2}y = 0,$$

and see the y' coefficient is $\frac{1}{x}$, which is allowed, while the singularity of the y coefficient is at worst like $\frac{1}{x^2}$, which is also allowed.

To find the indicial equation, rewrite

$$x^2y'' + x(1)y' + \left(-\frac{4}{9} + x^2\right)y = 0,$$

and $p_0 = 1$, $q_0 = -\frac{4}{9}$. So the indicial equation is

$$0 = r(r-1) + 1 \cdot r - \frac{4}{9} = r^2 - \frac{4}{9} = \left(r - \frac{2}{3}\right)\left(r + \frac{2}{3}\right).$$

So the exponents are $r_1 = \frac{2}{3}$ and $r_2 = -\frac{2}{3}$.

To set up the equation, write for $x > 0$

$$\begin{aligned} y &= x^r(a_0 + a_1x + \dots + a_nx^n + \dots) \\ &= a_0x^r + a_1x^{r+1} + \dots + a_nx^{r+n} + \dots, \\ y' &= ra_0x^{r-1} + (r+1)a_1x^r + \dots + (r+n)a_nx^{r+n-1} + \dots, \\ y'' &= r(r-1)a_0x^{r-2} + (r+1)ra_1x^{r-1} + \dots + (r+n)(r+n-1)a_nx^{r+n-2} + \dots \end{aligned}$$

The equation $x^2y'' + xy' + x^2y - \frac{4}{9}y = 0$ gives then the following

$$\begin{array}{cccccccc} 0 = & r(r-1)a_0x^r & + & (r+1)ra_1x^{r+1} & + \dots + & (r+n)(r+n-1)a_nx^{r+n} & + \dots \\ & + & ra_0x^r & + & (r+1)a_1x^{r+1} & + \dots + & (r+n)a_nx^{r+n} & + \dots \\ & & & & & + \dots + & a_{n-2}x^{r+n} & + \dots \\ & - & \frac{4}{9}a_0x^r & - & \frac{4}{9}a_1x^{r+1} & - \dots - & \frac{4}{9}a_nx^{r+n} & - \dots \end{array}$$

The x^r terms recover the indicial equation $r(r-1)a_0 + ra_0 - \frac{4}{9}a_0 = 0$, and so $r = \pm\frac{2}{3}$ with no restriction on a_0 . The x^{r+1} terms give $(r+1)ra_1 + (r+1)a_1 - \frac{4}{9}a_1 = 0$, and so $a_1 = 0$. The x^{n+r} terms give the recurrence relation

$$(r+n)(r+n-1)a_n + (r+n)a_n + a_{n-2} - \frac{4}{9}a_n = 0,$$

which implies

$$a_n = -\frac{a_{n-2}}{(r+n)(r+n-1) + (r+n) - \frac{4}{9}}.$$

For $r = \frac{2}{3}$, let $a_0 = c_1$, and we find $a_1 = 0$. To find the first three nonzero terms, compute

$$\begin{aligned} a_2 &= -\frac{a_0}{\left(\frac{2}{3}+2\right)\left(\frac{2}{3}+1\right) + \left(\frac{2}{3}+2\right) - \frac{4}{9}} = -\frac{3}{20}c_1, \\ a_3 &= -\frac{a_1}{\left(\frac{2}{3}+3\right)\left(\frac{2}{3}+2\right) + \left(\frac{2}{3}+3\right) - \frac{4}{9}} = 0, \\ a_4 &= -\frac{a_2}{\left(\frac{2}{3}+4\right)\left(\frac{2}{3}+3\right) + \left(\frac{2}{3}+4\right) - \frac{4}{9}} = \frac{9}{1280}c_1 \end{aligned}$$

On the other hand, for $r = -\frac{2}{3}$, let $a_0 = c_2$ and we have $a_1 = 0$ as above. Then

$$\begin{aligned} a_2 &= -\frac{a_0}{\left(-\frac{2}{3}+2\right)\left(-\frac{2}{3}+1\right) + \left(-\frac{2}{3}+2\right) - \frac{4}{9}} = -\frac{3}{4}c_2, \\ a_3 &= -\frac{a_1}{\left(-\frac{2}{3}+3\right)\left(-\frac{2}{3}+2\right) + \left(-\frac{2}{3}+3\right) - \frac{4}{9}} = 0, \\ a_4 &= -\frac{a_2}{\left(-\frac{2}{3}+4\right)\left(-\frac{2}{3}+3\right) + \left(-\frac{2}{3}+4\right) - \frac{4}{9}} = \frac{9}{128}c_2 \end{aligned}$$

All together, the general solution is

$$y = c_1|x|^{\frac{2}{3}} \left(1 - \frac{3}{20}x^2 + \frac{9}{1280}x^4 + \dots\right) + c_2|x|^{-\frac{2}{3}} \left(1 - \frac{3}{4}x^2 + \frac{9}{128}x^4 + \dots\right)$$