SOLUTION

1. (30 pts) Compute the general solution to each of the following equations. Show your work.

(a) \( y'' + 2y' + y = 0. \)
   Solution: If \( y = e^{rt} \), the characteristic equation is \( r^2 + 2r + 1 = 0 \). This factors to \((r + 1)^2 = 0\), and so \(-1\) is a double root. So the general solution is \( c_1 e^{-t} + c_2 te^{-t}. \)

(b) \( y'' + 2y' + 2y = 0. \)
   Solution: For \( y = e^{rt} \), we have \( r^2 + 2r + 2 = 0 \), with solutions
   \[ r = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = -1 \pm i, \]
   and so the general solution is \( c_1 e^{-t} \cos t + c_2 e^{-t} \sin t. \)

(c) \( x^2y'' - 6xy' + 12y = 0 \) (for \( x > 0 \)).
   Solution: For \( y = x^r \), we have the indicial
   \[ 0 = r(r - 1) - 6r + 12 = r^2 - 7r + 12 = (r - 3)(r - 4). \]
   So the roots are \( r = 3, 4 \), and the general solution (for \( x > 0 \)) is \( c_1 x^3 + c_2 x^4. \)

2. (10 pts) Find the first five terms of the power series solution (up to the \( a_4x^4 \) term) around \( x = 0 \) to the initial value problem
   \[ y'' + (2 + x)y' - y = x^2 + 5, \quad y(0) = 1, \quad y'(0) = 3. \]
   Show your work.
   Solution: Compute for
   \[
   \begin{align*}
   y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\
   y' &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\
   y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots 
   \end{align*}
   \]
It helps to rewrite our equation as $y'' + 2y' + xy' - y = x^2 + 5$, which becomes

\[
\begin{align*}
2a_2 & + 6a_3x + 12a_4x^2 + \cdots \\
+ 2a_1 & + 4a_2x + 6a_3x^2 + \cdots \\
+ & a_1x + 2a_2x^2 + \cdots \\
- a_0 & - a_1x - a_2x^2 - \cdots \\
= & 5 + x^2 
\end{align*}
\]

The initial conditions imply $a_0 = y(0) = 1$ and $a_1 = y'(0) = 3$. Then the constant terms imply

\[
2a_2 + 2a_1 - a_0 = 5, \quad 2a_2 + 2(3) - 1 = 5, \quad a_2 = 0.
\]

The $x$ terms imply

\[
6a_3 + 4a_2 + a_1 - a_1 = 0, \quad 6a_3 + 4(0) = 0, \quad a_3 = 0.
\]

The $x^2$ terms imply

\[
12a_4 + 6a_3 + 2a_2 - a_2 = 1, \quad 12a_4 + 6(0) + 0 = 1, \quad a_4 = \frac{1}{12}.
\]

Thus the solution

\[
y \sim 1 + 3x + \frac{1}{12} x^4.
\]

3. (10 pts) Find the recurrence relation for the coefficients of power series solution around $x = 0$ to differential equation

\[y'' + x^2y = 0.\]

Show your work.

**Solution:** Compute

\[
\begin{align*}
y & = a_0 + a_1x + a_2x^2 + \cdots a_nx^n + \cdots \\
y'' & = 2a_2 + \cdots + n(n-1)a_nx^{n-2} + \cdots \\
& = 2a_2 + \cdots + (n+2)(n+1)a_{n+2}x^n + \cdots \\
x^2y & = a_0x^2 + \cdots + a_nx^{n+2} + \cdots \\
& = a_0x^2 + \cdots + a_{n-2}x^n + \cdots 
\end{align*}
\]

So, comparing $x^n$ terms, $y'' + x^2y = 0$ implies $(n + 2)(n + 1)a_{n+2} + a_{n-2} = 0$, and so the recurrence relation is

\[a_{n+2} = -\frac{a_{n-2}}{(n + 2)(n + 1)}.\]

4. (10 pts)

(a) Use the method of undetermined coefficients to find a solution to

\[y''' - 4y' = e^x.\]
Show your work.

**Solution:** Let the solution \( Y = Ae^t \), and compute \( Y' = Ae^t, \ Y'' = Ae^t, \ Y''' = Ae^t \). Plug in to find

\[
Ae^t - 4Ae^t = e^t, \quad -3A = 1, \quad A = -\frac{1}{3},
\]

and the solution is \( Y = -\frac{1}{3}e^t \).

(b) Find the general solution to \( y''' - 4y' = e^t \).

Show your work.

**Solution:** The general solution to this nonhomogeneous equation is the particular solution from part (a) plus the general solution to the homogeneous equation \( y''' - 4y' = 0 \). For this, let \( y = e^{rt} \), and find the characteristic equation

\[
0 = r^3 - 4r = r(r - 2)r + 2, \quad r = 0, 2, -2.
\]

So the general solution to the homogeneous equation is

\[
c_1 + c_2e^{2t} + c_3e^{-2t},
\]

and the general solution to our problem is

\[
-\frac{1}{3}e^t + c_1 + c_2e^{2t} + c_3e^{-2t}.
\]

5. (5 pts) Use Abel’s Theorem to determine the Wronskian of two solutions to the equation

\[
xy'' + x^2y' + y = 0.
\]

**Solution:** Put the equation in the standard format

\[
y'' + xy' + x^{-1}y = 0.
\]

for \( p(x) = x \). Abel’s Theorem then states that the Wronskian for any two solutions is equal to

\[
c \exp \left( -\int p \, dx \right) = c \exp \left( -\int x \, dx \right) = ce^{-\frac{1}{2}x^2}
\]

for a constant \( c \).