HINTS: PROBLEM SET 7

- §3.2, # 2.
  - These should follow easily by the rules for determinant.
- §3.2, # 13.
  - Note we assume that the compact hypersurface H has no boundary.
  - Use the Jordan-Brouwer separation theorem to show H is the boundary of a domain $D \subset \mathbb{R}^n$. Is D naturally an oriented manifold?
- §3.3, # 4.
  - For part (a), show that every point in the circle is a regular point of $f/|f|$.
- §3.3, # 8.
  - We essentially did this construction in class. Show that the map f determined by g is homotopic to the map determined by $g_1(t) = g(0) + tq$. The key point is to make sure the maps are all well-defined as maps from S^1 to S^1.
- §3.5, # 3.
  - Consider a coordinate chart $\phi\colon U \to X$, where U is an open subset of $\mathbb{R}^k$ and $\phi(0) = x$. Let $t^1, \ldots, t^n$ be coordinates on $\mathbb{R}^n$. Let $f\colon X \to \mathbb{R}^N$ be the immersion in the problem. Show that the tangent space to X at a point $\phi(t) \in \phi(U)$ is the span of the n vectors
    $$\frac{\partial(f \circ \phi)}{\partial t^i}(t).$$
  - In the neighborhood $\phi(U)$ of x, show that a smooth vector field $\vec{v}\colon y \mapsto \vec{v}(y) \in T_y(X) \subset \mathbb{R}^N$ can be described by
    $$\vec{v}(\phi(t)) = \sum_{i=1}^{n} a_i(t) \frac{\partial(f \circ \phi)}{\partial t^i}(t)$$
    for $a_1, \ldots, a_n$ smooth functions of $t = (t^1, \ldots, t^n)$.
  - Use the previous formulation, along with the product rule, to conclude
    $$\frac{\partial(\vec{v} \circ \phi)}{\partial t^i}(0) \in T_x(X)$$
    for $i = 1, \ldots, n$.

- §3.6, # 10.
  - Use the result of §3.5, # 7 (you don't have to prove it) to show that around any isolated zero of $\vec{v}$, there is a vector field $\vec{v}_1$ equal to $\vec{v}$ outside a small neighborhood of the zero of $\vec{v}$ so that $\vec{v}_1$ only has nondegenerate zeros. This neighborhood may be made arbitrarily small.
  - Let x be an isolated zero of $\vec{v}$. Compute $\mathrm{ind_x \tilde{v}}$ as the degree of
  $$\frac{\vec{v}}{|\vec{v}|} : S_\epsilon^{\ell-1} \to S^{\ell-1}$$
  (Here of course $S_\epsilon^{\ell-1}$ is the boundary of the ball B_ϵ of radius ϵ centered at x.) Prove that this is the same as the degree of $\vec{v}_1/|\vec{v}_1|$. (Let $\vec{v}$ and $\vec{v}_1$ differ only inside B_ϵ.)
  - $\vec{v}_1$ has only nondegenerate zeros x_i inside $S_\epsilon^{\ell-1}$. Choose small balls B_i centered at the x_i so that all their closures are disjoint and all $\overline{B_i} \subset B_\epsilon$. Show that the degree of
  $$\frac{\vec{v}_1}{|\vec{v}_1|} : S_\epsilon^{\ell-1} \to S^{\ell-1}$$
  is equal to the sum
  $$\sum_i \mathrm{ind}_{x_i} \vec{v}_1.$$
  - Reduce to the case that all the zeros of $\vec{v}$ are nondegenerate.
  - Apply the technique we used for number 5 in class. (Note that a vector field on an open set $U \subset \mathbb{R}^k$ is equivalent to a map from U to $\mathbb{R}^k$.)