BUMP FUNCTIONS

Proposition 1. *Define*

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

$f(x)$ is smooth. In particular, $f^{(n)}(0) = 0$ for all $n \geq 0$.

Proof. It is clear that $f(x)$ is smooth except at $x = 0$. So we just need to verify that $f^{(n)}(0) = 0$ for all $n \geq 0$. (Note that if we prove that $f^{(n)}(0) = 0$, then that implies already that $f^{(n-1)}(x)$ is continuous at $x = 0$. It is obvious that $f^{(n-1)}(x)$ is continuous away from $x = 0$. So if we prove all the derivatives exist, we automatically get that they are continuous, which is part of the definition of smoothness.)

First of all, we check that $f'(0) = 0$. It is clear that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{0}{x} = 0.$$

The other one-sided limit

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{x} \\ &= \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{e^{\frac{1}{x}}} \\ &= \lim_{y \to \infty} \frac{y}{e^y} \\ &\overset{\ell}{=} \lim_{y \to \infty} \frac{1}{e^y} \\ &= 0 \end{aligned}$$

Here we've substituted $y = \frac{1}{x}$, and $\overset{\ell}{=}$ indicates the use of l'Hôpital's rule. Since the left and right hand limits are equal, we have $f'(0) = 0$.

For higher derivatives, compute for $x > 0$, $f'(x) = \frac{1}{x^2} e^{-\frac{1}{x}}$. It is easy to show by induction that all higher derivatives have a similar form: for $x > 0$,

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x}}$$

for P_n a polynomial function.

Proof. We've checked that for $n = 1$, $x > 0$, $f'(x)$ is of the form $P_1(\frac{1}{x})e^{\frac{1}{x}}$ for $P_1(y) = y^2$. Now assume that $f^{(n)}(x) = P_n\left(\frac{1}{x}\right)e^{-\frac{1}{x}}$ for $x > 0$. Then compute

$$\begin{aligned} f^{(n+1)}(x) &= P_n'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)e^{-\frac{1}{x}} + P_n\left(\frac{1}{x}\right)e^{-\frac{1}{x}}\left(\frac{1}{x^2}\right) \\ &= P_{n+1}\left(\frac{1}{x}\right)e^{-\frac{1}{x}} \end{aligned}$$

for $P_{n+1}(y) = -y^2P_n'(y) + y^2P_n(y)$, which is a polynomial whenever P_n is. So we've checked that $f'(x) = f^{(1)}(x)$ satisfies the hypothesis, and also that if the hypothesis is true for $f^{(n)}(x)$, then it's true for $f^{(n+1)}(x)$. By induction, the hypothesis is true for all positive integers n. □

We use induction again to show that $f^{(n)}(0) = 0$. We've checked the case $n = 1$ above. Now assume that $f^{(n)}(0) = 0$. To compute $f^{(n+1)}(0)$, the left-hand limit is 0 as above, and the right-hand limit

$$\begin{aligned} \lim_{x\to 0^+}\frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} &= \lim_{x\to 0^+}\frac{f^{(n)}(x) - 0}{x} \\ &= \lim_{x\to 0^+}\frac{1}{x}P_n\left(\frac{1}{x}\right)e^{-\frac{1}{x}} \\ &= \lim_{x\to 0^+}\frac{\tilde{P}_n\left(\frac{1}{x}\right)}{e^{\frac{1}{x}}} \\ &= \lim_{y\to\infty}\frac{\tilde{P}_n(y)}{e^y} \\ &= 0. \end{aligned}$$

Here $y = \frac{1}{x}$, and $\tilde{P}_n(y) = yP_n(y)$ is another polynomial. The last limit is a standard fact that can by proved by l'Hôpital's rule and induction. Since the left and right hand limits are both 0, we have $f^{(n+1)}(0) = 0$, which completes the induction. □

Proposition 2. *Given $a < b$, there is a smooth function $h(x)$ so that $h(x) = 0$ for $x \leq a$, $0 < h(x) < 1$ for $x \in (a, b)$, and $h(x) = 1$ for $x \geq b$.*

Proof. First let $g(x) = f(x-a)f(b-x)$ for f defined above. It is clear g is smooth, nonnegative, and $g(x) > 0 \iff x \in (a, b)$. Then define

$$h(x) = \frac{\int_{-\infty}^{x} g(t)\,dt}{\int_{-\infty}^{\infty} g(t)\,dt}.$$

It is easy to check that $h(x)$ has the relevant properties. □

Proposition 3. *Let $0 < a < b$. Then there is a smooth real-valued function $i(z)$ on $\mathbb{R}^k$ so that $i(z) = 1$ if $|z| \leq a$, $0 < i(z) < 1$ for $|z| \in (a, b)$, and $i(z) = 0$ for $|z| \geq b$.*

Proof. Let $\tilde{h}(x) = 1 - h(x)$ for $h(x)$ above. Then it is easy to check that $i(z) = \tilde{h}(|z|)$ satisfies the relevant properties. In particular, the chain rule shows that $i(z)$ is smooth except possibly at $z = 0$. But in a neighborhood of $z = 0$, $i(z)$ is identically 1, so it is smooth there as well. □

Remark. Notice that the graph of $i(z)$ looks like a bump. It is called a *bump function*.