MANIFOLDS AND CONNECTEDNESS

Proposition 1. *Let X be a topological manifold. Then X is locally connected. In other words, for every $x \in X$, $x \in U$, U open in X, there is a connected open set V so that $x \in V \subset U$.*

Remark. This is not true for all topological spaces Y. An example is $Y = \{0, 1, 1/2, 1/3, 1/4, \dots\}$ with the subspace topology from $Y \subset \mathbb{R}$.

Proof. Homeomorphic images of small open balls in $\mathbb{R}^k$ form a basis for the topology of X. (Each such homeomorphism is the restriction of a local parametrization of X.) □

Proposition 2. *Let X be a smooth manifold. Then X is connected if and only if every $x, y \in X$ can be joined by a smooth path $\gamma\colon [0,1] \to X$, so that $\gamma(0) = x$, $\gamma(1) = y$.*

Proof. **Step 1:** Assume every $x, y \in X$ can be joined by a smooth path. Then in particular, X is path-connected. This implies X is connected (for X any topological space).

Step 2: Assume X is connected. Choose $x \in X$. Let

$$E = \{y \in X : x \text{ and } y \text{ can be joined by a smooth path}\}.$$

We need to show $E = X$. So since X is connected, we must show E is nonempty, open, and closed. E is nonempty since $x \in E$.

To show E is closed, let $y_i \in E$, with $\lim y_i = y$. Then we want $y \in E$. (Note that Whitney's embedding theorem shows that X is a metrizable topological space.) Choose a coordinate neighborhood U of y which is diffeomorphic to an open ball B. Then there is an N so that if $i \geq N$, $y_i \in U$. So x and y_N can be connected by a smooth path, and moreover, y_N and y can be connected by a smooth path (take a straight line segment in B). These can be joined together to make a smooth path from x to y (use the same technique as in the proof that homotopy is an equivalence relation). Therefore, $y \in E$ and E is closed.

To show E is open, let $y \in E$. Then consider an open neighborhood U of y which is diffeomorphic to a ball B. Then any $z \in U$ can be connected to y by a smooth path (a straight line segment in B will do). Therefore, as above, x and z can be connected by a smooth path. Therefore $U \subset E$ and E is open. □