PROBLEM SET 4–DUE WED., OCTOBER 29

- Guillemin & Pollack, Section 1.6, Problems 1, 2.
- Guillemin & Pollack, Section 1.7, Problems 5, 6.
- Let S be the topological space $\mathbb{C}^2 \setminus \{0\}$. The topology on $\mathbb{C}^2$ is the topology of $\mathbb{R}^4$. In particular, natural real coordinates on $\mathbb{C}^2$ are x^1, y^1, x^2, y^2. The complex valued functions $z^j = x^j + iy^j$ for $j = 1, 2$, where $i = \sqrt{-1}$, are the natural complex coordinates on $\mathbb{C}^2$. Define an equivalence relation on S, by saying $p \sim q$ if $p = \alpha q$ for an nonzero *complex* number α. Note that $p = (p^1, p^2)$, $q = (q^1, q^2)$ for p^j, q^j complex numbers, and $\alpha q = (\alpha q^1, \alpha q^2)$ is the usual multiplication of a complex vector by a complex scalar. Define $\mathbb{CP}^1$ to the the quotient topological space $S/\sim$. Follow this outline to show that $\mathbb{CP}^1$ is naturally a compact 2-dimensional manifold. Use the discussion about $\mathbb{RP}^n$ as a model (see the notes on the real definition of a smooth manifold).
  (1) Show $\mathbb{CP}^1$ is Hausdorff. This may require a little ingenuity to do from scratch. Or you can show it after you have established the coordinates below.
  (2) Show $\mathbb{CP}^1$ is compact. In particular, recognize that $\mathbb{CP}^1 = \pi(S^3)$, where π is the projection from $\mathbb{C}^2 \setminus \{0\}$ to $\mathbb{CP}^1$ and S^3 is the unit sphere in $\mathbb{C}^2 \equiv \mathbb{R}^4$.
  (3) For $j = 1, 2$, let $P_j = \{z^j = 1\} \subset \mathbb{C}^2$. Show that $\pi : P_j \to \mathbb{CP}^1$ is a homeomorphism onto its image. Let $U_j = \pi(P_j)$. Show that $\mathbb{CP}^1 = U_1 \cup U_2$ is an open cover.
  (4) Find natural maps $\chi_j : \mathbb{R}^2 \to P_j$ for each $j = 1, 2$. These χ_j should be diffeomorphisms.
  (5) Define $\phi_j = \pi \circ \chi_j$ and show the $\{(\mathbb{R}^2, \phi_j)\}_{j=1,2}$ is an atlas of $\mathbb{CP}^1$.
  (6) Explicitly compute the gluing map $\phi_1^{-1} \circ \phi_2$ and the domain $\phi_2^{-1}(U_1 \cap U_2)$. Check that this gluing map is smooth. (Remember that if $z = x + iy$, $\frac{1}{z} = \frac{x-iy}{x^2+y^2}$.) Can this case represent all gluing maps?
  (7) Conclude that $\mathbb{CP}^1$ is a smooth manifold of dimension 2.