THE REAL DEFINITION OF A SMOOTH MANIFOLD

1. Topological manifolds

A topological space X is called *sigma-compact* if X is equal to a countable union of compact subsets. In other words, there are compact subsets K_i of X for $i = 1, 2, 3, \dots$ so that

$$X = \bigcup_{i=1}^{\infty} K_i.$$

Recall a topological space X is Hausdorff if for every $x, y \in X$, $x \neq y$, there are open neighborhoods $\mathcal{O}_x \ni x$, $\mathcal{O}_y \ni y$ so that $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$.

A *topological manifold* of dimension k is a sigma-compact, Hausdorff topological space X which is locally homeomorphic to $\mathbb{R}^k$. In other words, around every $x \in X$, there is a neighborhood W which is homeomorphic to an open set $U \subset \mathbb{R}^k$. The homeomorphism $\phi \colon U \to W$ is as usual called a *parametrization* of the neighborhood W of x. ϕ^{-1} is called a *local coordinate system* on the neighborhood X near x.

Proposition 1. *Any submanifold X of $\mathbb{R}^N$ is Hausdorff and sigma-compact.*

Proof. That X is Hausdorff follows from the fact that X is a metric space with the induced metric from $\mathbb{R}^N$.

To prove that X is sigma-compact, define $bX = \overline{X} \setminus X$. (We can think of bX as the boundary of the manifold X.) Then if $bX \neq \emptyset$, let

$$K_n = \{x \in X : |x| \leq n,\ \operatorname{dist}(x, bX) \geq \tfrac{1}{n}\}.$$

Here $\operatorname{dist}(x, bX) = \inf\{|x - y| : y \in bX\}$. In the case $bX = \emptyset$, then define

$$K_n = \{x \in X : |x| \leq n\}.$$

It is an exercise to show that $X = \bigcup_{n=1}^{\infty} K_n$, and that each K_n is closed and bounded in $\mathbb{R}^N$ (and is thus compact). □

2. Smooth manifolds

So far we cannot do any calculus on our manifold X, since it is just a topological space (and not necessarily a subset of $\mathbb{R}^N$, as we've considered before). So we cannot talk about tangent spaces, immersions, transversal intersections, etc., yet.

For starters, let's consider real valued functions on X. So $f\colon X \to \mathbb{R}$. What should it mean for such a function to be smooth? Certainly we want the corresponding function to be smooth in each coordinate chart U. In other words, if $\phi: U \to X$ is a local parametrization, then we want $f \circ \phi\colon U \to \mathbb{R}$ to be smooth. Note that this makes sense because U is an open subset of $\mathbb{R}^k$.

There is a potential problem, however. A given point $x \in X$ is typically contained in more than one coordinate chart $\phi(U) \subset X$. We must make sure that our notion of smoothness is independent of the coordinate chart we choose. So consider the case when $x \in \phi_1(U_1) \cap \phi_2(U_2)$, where for $\alpha = 1, 2$, U_α is an open subset of $\mathbb{R}^k$ and $\phi_\alpha\colon U_\alpha \to X$ is a coordinate parametrization. Then a given $f\colon X \to \mathbb{R}$ is smooth if each of the $f \circ \phi_\alpha$ is smooth. Note each ϕ_α is a homeomorphism onto its image. Now let $W_\alpha = \phi_\alpha(U_\alpha)$. Each W_α is a neighborhood of x, and let $W_{12} = W_1 \cap W_2$ be the intersection. Then, on the restricted domains $\phi_\alpha^{-1}(W_{12})$ (i.e. we're shrinking each U_α)

$$\phi_\alpha\colon \phi_\alpha^{-1}(W_{12}) \to W_{12}$$

is a homeomorphism. Now let's assume $f \circ \phi_1$ is smooth. Then on restricted domain at least,

$$f \circ \phi_2 = (f \circ \phi_1) \circ (\phi_1^{-1} \circ \phi_2).$$

Then it is possible to differentiate $f \circ \phi_2$, as long as $\phi_1^{-1} \circ \phi_2$ is differentiable (by the chain rule). And if we want to compute higher derivatives, we'll need all the higher derivatives of $\phi_1^{-1} \circ \phi_2$ as well. The condition we require is that

$$\phi_1^{-1} \circ \phi_2\colon \phi_2^{-1}(W_{12}) \to \phi_1^{-1}(W_{12})$$

be a diffeomorphism. Note that since the domain and range of this function are open subsets of $\mathbb{R}^k$, it makes sense to talk about $\phi_1^{-1} \circ \phi_2$ being differentiable. Also, our function is already a homeomorphism by the fact X is a topological manifold: what we need in addition is smoothness and smoothness of the inverse.

Before we define smooth manifolds, one more definition is useful: an *atlas* of a manifold X of dimension k is a collection of parametrizations $\{(U_\alpha, \phi_\alpha)\}$—i.e. $U_\alpha \subset \mathbb{R}^k$ is an open subset and $\phi_\alpha : U_\alpha \to X$ is a homeomorphism onto its image W_α—so that

$$X = \bigcup_\alpha \phi_\alpha(U_\alpha).$$

In other words, the coordinate charts $W_\alpha = \phi_\alpha(U_\alpha)$ form an open cover of X.

A *smooth manifold* X is a topological manifold together with an atlas (U_α, ϕ_α) so that each

$$\phi_\alpha^{-1} \circ \phi_\beta$$

is a diffeomorphism. Each such function $\phi_\alpha^{-1} \circ \phi_\beta$ is called a *gluing map.* If the two coordinate charts $W_\alpha \cap W_\beta = \emptyset$, then this condition is vacuous. (As above, $W_\alpha = \phi_\alpha(U_\alpha)$.) In general, as above, the domain and range of $\phi_\alpha^{-1} \circ \phi_\beta$ are open subsets of $\mathbb{R}^k$. The domain is $\phi_\beta^{-1}(W_{\alpha\beta})$, and the range is $\phi_\alpha^{-1}(W_{\alpha\beta})$, where $W_{\alpha\beta} = W_\alpha \cap W_\beta$. In this case when each gluing map $\phi_\alpha^{-1} \circ \phi_\beta$ is a diffeomorphism, the atlas (U_α, ϕ_α) is called a *smooth atlas* of X.

A useful way of thinking of a manifold with an atlas is as a union of the coordinate charts glued, or patched, together according to the functions $\phi_\alpha^{-1} \circ \phi_\beta$. Our manifold X can be identified as a set with the union of the coordinate neighborhoods U_α modulo patching together. More specifically, for our atlas (U_α, ϕ_α), consider the disjoint union

$$\mathcal{U} = \bigsqcup_\alpha U_\alpha.$$

Then the patching together is performed by an equivalence relation. Two points $x \in U_\alpha$, $y \in U_\beta$ are equivalent if they map to the same point in our manifold X. In terms of the atlas, this just means that

$$x \sim y \qquad \iff \qquad x = (\phi_\alpha^{-1} \circ \phi_\beta)(y).$$

Then as a set at least X is equal to the quotient $\mathcal{U}/\sim$.

If X and Y are smooth manifolds of respective dimensions k and ℓ, then a map $f : X \to Y$ is *smooth* if it is smooth on every coordinate chart. In other words, for all $x \in X$, if $y = f(x)$, then we can find coordinate charts in our smooth atlases so that f is a smooth function in these coordinates. More explicitly, if $x \in \phi(U)$, $y \in \psi(V)$, for smooth parametrizations ϕ and ψ, then we require

$$\psi^{-1} \circ f \circ \phi$$

to be smooth. Note the domain of this function is $U \cap \phi^{-1}(\psi(V))$ (so we've had to shrink U a little). This definition of f smooth is independent of the coordinates chosen by the fact that the atlases are smooth.

3. Tangent vectors

For a nice description of tangent vectors and the tangent bundle for manifolds, see the book by Warner, *Foundations of Differentiable Manifolds and Lie Groups.*

In particular, the tangent space T_xX of a smooth manifold at a point x can be defined independently of any inclusion of $X \to \mathbb{R}^N$ as a submanifold.

4. An exhaustion of any manifold

Let X be a topological space. A countable collection of open subsets $\{\mathcal{O}_i\}_{i=1}^{\infty}$ is an *exhaustion* of X if

- $X = \displaystyle\bigcup_{i=1}^{\infty} \mathcal{O}_i$, and
- $\overline{\mathcal{O}_i}$ is compact and $\overline{\mathcal{O}_i} \subset \mathcal{O}_{i+1}$.

(The last pair of conditions is denoted $\mathcal{O}_i \subset\subset \mathcal{O}_{i+1}$.)

Proposition 2. *Any manifold X has an exhaustion by open subsets.*

Proof. Since X is sigma-compact,

$$X = \bigcup_{i=1}^{\infty} K_i, \qquad K_i \text{ compact.}$$

An easy lemma we will use repeatedly is this:

Lemma 3. *A finite union of compact subsets of a topological space is compact.*

The proof is left as an exercise.

Let $\mathcal{O}_0 = \emptyset$. We prove the Proposition by induction on the following statement for $n > 0$: There are open sets $\mathcal{O}_i$ for $i = 1, \dots, n$ so that

- $\mathcal{O}_i \supset K_i \cup \overline{\mathcal{O}_{i-1}}$.
- $\overline{\mathcal{O}_i}$ is compact.

To complete the initial $n = 1$ step of the induction, we need only construct $\mathcal{O}_1$. Define $\tilde{K}_1 = K_1$. For all $x \in \tilde{K}_1$, there is a coordinate neighborhood $W = \phi(U)$, $U \subset \mathbb{R}^k$ open. We may assume $\phi(0) = x$. Then let N_x be a neighborhood of x, so that $N_x \subset\subset W$. (This is possible since we can always find a small ball $\phi(B_\epsilon(0)) \subset W$. Then take $N_x = \phi(B_{\epsilon/2}(0))$. That $N_x \subset\subset W$ follows from Proposition 4 below.) Now $\{N_x\}_{x\in\tilde{K}_1}$ is an open cover of $\tilde{K}_1$, and thus has a finite subcover $\{N_{x_i}\}_{i=1}^{M}$. We let

$$\mathcal{O}_1 = \bigcup_{i=1}^{M} N_{x_i}.$$

Then by the lemma above, $\overline{\mathcal{O}_1} = \bigcup_{i=1}^{M} \overline{N_{x_i}}$ is compact.

Now assume by induction, we have open sets $\mathcal{O}_1, \dots, \mathcal{O}_n$ satisfying the inductive criteria: So in particular, $\mathcal{O}_n$ is compact and $\mathcal{O}_n \supset$

$K_n \cup \overline{\mathcal{O}_{n-1}}$. Now let $\tilde{K}_{n+1} = K_{n+1} \cup \overline{\mathcal{O}_n}$. By the lemma above and the induction hypothesis, $\tilde{K}_{n+1}$ is compact. Now we can apply the argument of the previous paragraph to define

$$\mathcal{O}_{n+1} = \bigcup_{i=1}^{M} N_{x_i},$$

for $\{N_{x_i}\}_{i=1}^M$ a finite open cover of $\tilde{K}_{n+1}$ so that each $\overline{N_{x_i}}$ is compact. It is easy to check the induction step now.

Since each $\mathcal{O}_n \supset K_n$, $X = \bigcup_{n=1}^\infty \mathcal{O}_n$. Also, by induction, $\mathcal{O}_n \subset\subset \mathcal{O}_{n+1}$. □

5. The Hausdorff property and manifolds

In this section, we drop temporarily the assumption that manifolds are Hausdorff in order to discuss a useful property which is equivalent to the Hausdorff property for manifolds.

Proposition 4. *Let X be a manifold of dimension n. The following property is equivalent to X being Hausdorff:*

For every local parametrization given by (U, ϕ)—in other words, $U \subset \mathbb{R}^n$ is open and $\phi\colon U \to X$ is a homeomorphism onto its image—and for every open $V \subset U$ so that $\overline{V} \subset U$ is compact, then $\overline{\phi(V)} = \phi(\overline{V})$.

Proof. First of all, let us assume this property and show X is Hausdorff. Let $x \neq y$ be points in X. If x and y are in a single coordinate chart $\phi(U)$, then since U is open, there are small open balls B_x and B_y centered at $\phi^{-1}(x)$ and $\phi^{-1}(y)$ contained in $U \in \mathbb{R}^n$. Then choose the radii of these balls to be strictly less than $\frac{1}{2}|\phi^{-1}(x) - \phi^{-1}(y)|$ to guarantee they are disjoint (by the triangle inequality). Then $\phi(B_x)$ and $\phi(B_y)$ are the required disjoint open sets in X.

If $x \neq y$ are not in a single coordinate chart, then let $x \in \phi(U)$, $y \notin \phi(U)$. Then choose a small open ball V centered at $\phi^{-1}(x)$ in $U \subset \mathbb{R}^n$ so that $\overline{V} \subset U$. Then since $\overline{\phi(V)} = \phi(\overline{V}) \subset \phi(U)$, $y \in X \backslash \overline{\phi(V)}$, which is an open set disjoint from the open neighborhood $\phi(V)$ of x. Therefore X is Hausdorff.

Now assume X is Hausdorff. Let $U \subset \mathbb{R}^n$ be a coordinate chart with parametrization ϕ and an open subset $V \in U$ so that $\overline{V} \subset U$ is compact. We want to show that $\overline{\phi(V)} = \phi(\overline{V})$.

First we show $\overline{\phi(V)} \supset \phi(\overline{V})$. This just uses the fact ϕ is continuous: If W is an open subset of X which does not intersect $\phi(V)$, then $\phi^{-1}(W)$ is an open subset of U which does not intersect V. Thus $\phi^{-1}(W) \cap \overline{V} = \emptyset$ by the definition of $\overline{V}$. Therefore, $W \cap \phi(\overline{V}) = \emptyset$. Since W is contained in the complement of $\overline{\phi(V)}$, then $\overline{\phi(V)} \supset \phi(\overline{V})$.

To show $\overline{\phi(V)} = \phi(\overline{V})$, it suffices to show $\phi(\overline{V})$ is closed. Since $\overline{V}$ is compact and ϕ is continuous, $\phi(\overline{V})$ is compact. And since X is Hausdorff, $\phi(\overline{V})$ is closed. □

Remark. Notice that the Hausdorff property of manifolds then implies that they are locally compact. In other words, every $x \in X$ has a neighborhood W with compact closure.

6. An example

Real projective space $\mathbb{RP}^n$ is an n-dimensional smooth manifold which is not be naturally defined as a subset of $\mathbb{R}^N$. Instead, the definition is in terms of the quotient topology. Consider the set $S = \mathbb{R}^{n+1} \setminus \{0\}$. We define an equivalence relation on S. If $p, q \in S$, we say $p \sim q$ if there is a nonzero real number γ such that $p = \gamma q$. Notice that $p \sim q$ if and only if p and q are on the same line through the origin. Form the quotient space $\mathbb{RP}^n = S/\sim$, and give $\mathbb{RP}^n$ the quotient topology. Thus $\mathbb{RP}^n$ is a naturally the set of lines through the origin in $\mathbb{R}^{n+1}$. The quotient projection from S to $\mathbb{RP}^n$ is denoted by $\pi\colon p \mapsto [p]$.

We claim that $\mathbb{RP}^n$ is naturally a smooth manifold. First we check the topological properties.

Lemma 5. *$\mathbb{RP}^n$ is Hausdorff.*

Proof. A set $U \subset \mathbb{RP}^n$ is open if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. The set $\pi^{-1}(U)$ is a collection of lines, in other words a cone with the origin deleted. If $[p] \neq [q]$, then the two lines $\pi^{-1}(p)$ and $\pi^{-1}(q)$ are distinct lines through the origin. Notice that they don't meet in S. Let θ be the angle between these two lines. Then let

$$C_p = \bigcup_{\angle(\ell, \pi^{-1}(p)) < \theta/2} \ell \setminus \{0\}$$

Here ℓ represents any line through the origin in $\mathbb{R}^{n+1}$ which meets $\pi^{-1}(p)$ with angle less than $\theta/2$. Similarly define C_q. Then since $C_p \cap C_q = \emptyset$, and both are open cones in S, the sets $U_p = \pi(C_p)$ and $U_q = \pi(C_p)$ are disjoint neighborhoods of $[p], [q]$. □

Lemma 6. *$\mathbb{RP}^n$ is compact (and is therefore sigma-compact).*

Proof. $\mathbb{RP}^n = \pi(S^n)$, where S^n is the unit sphere in $\mathbb{R}^{n+1}$, which is compact. Then since π is continuous, $\mathbb{RP}^n$ is compact. □

To order to prove $\mathbb{RP}^n$ is a manifold, then we need to find a smooth atlas. There is a standard choice of $n+1$ coordinate parametrizations which make up an atlas of $\mathbb{RP}^n$. Define for each $i = 1, \ldots, n+1$,

$$P_i = \{x = (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i = 1\}.$$

It is obvious that each P_i is diffeomorphic to $\mathbb{R}^n$ (the functions $\{x^j\}_{j\neq i}$ are n coordinate functions). If $p = (p^1, \ldots, p^{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$, then at least one coordinate $p^i \neq 0$. Then $[p] = [\frac{1}{p^i}p]$, and $\frac{1}{p^i} \in P_i$. This shows that every $[p] \in \mathbb{RP}^n$ is contained in $\pi(P_i)$ for at least one P_i. In other words,

$$\mathbb{RP}^n = \bigcup_{i=1}^{n+1} \pi(P_i)$$

This is an open cover of $\mathbb{RP}^n$ since each

$$\pi^{-1}(\pi(P_i)) = \{x \in \mathbb{R}^{n+1} : x^i \neq 0\}$$

is open. Let $U_i = \pi(P_i)$. Note that $U_i = \{[p] : p^i \neq 0\}$. It is straightforward to check that $\pi\colon P_i \to U_i$ is a homeomorphism.

Now we must check that the for these coordinates, the gluing maps are diffeomorphisms. For each $i \in \{1, \ldots, n+1\}$, let

$$x_i = (x_i^1, \ldots, x_i^{i-1}, x_i^{i+1}, \ldots, x_i^{n+1})$$

be coordinates in $\mathbb{R}^n$. Also define a diffeomorphism $\chi_i : \mathbb{R}^n \to P_i$ by

$$\chi_i(x_i) = (x_i^1, \ldots, x_i^{i-1}, 1, x_i^{i+1}, \ldots, x_i^{n+1}).$$

Then let $\phi_i = \pi \circ \chi_i \colon \mathbb{R}^n \to U_i$. These are the coordinate parametrizations. We need to check that on $\phi_i^{-1}(U_i \cap U_k)$, $\phi_k^{-1} \circ \phi_i$ is smooth. Now for $[p] \in U_i \cap U_k$, $p^i \neq 0$, $p^k \neq 0$. Therefore,

$$\phi_i(x_i) \in U_i \cap U_k \iff x_i^k \neq 0.$$

Compute

$$\begin{aligned}
\phi_k^{-1}(\phi_i(x_i)) &= \phi_k^{-1}([x_i^1, \ldots, x_i^{i-1}, 1, x_i^{i+1}, \ldots, x_i^k, \ldots, x_i^{n+1}]) \\
&= \phi_k^{-1}\left(\left[\frac{x_i^1}{x_i^k}, \ldots, \frac{x_i^{i-1}}{x_i^k}, \frac{1}{x_i^k}, \frac{x_i^{i+1}}{x_i^k}, \ldots, 1, \ldots, \frac{x_i^{n+1}}{x_i^k}\right]\right) \\
&= \left(\frac{x_i^1}{x_i^k}, \ldots, \frac{x_i^{i-1}}{x_i^k}, \frac{1}{x_i^k}, \frac{x_i^{i+1}}{x_i^k}, \ldots, \frac{x_i^{k-1}}{x_i^k}, \frac{x_i^{k+1}}{x_i^k}, \ldots, \frac{x_i^{n+1}}{x_i^k}\right).
\end{aligned}$$

Since $x_i^k \neq 0$, then it's clear this function is smooth. The inverse is smooth since $(\phi_k^{-1} \circ \phi_i)^{-1} = \phi_i^{-1} \circ \phi_k$ and we may repeat the argument with i and k interchanged.