THE REAL DEFINITION OF A SMOOTH MANIFOLD

1. Topological manifolds

A topological space \( X \) is called \( \sigma \)-compact if \( X \) is equal to a countable union of compact subsets. In other words, there are compact subsets \( K_i \) of \( X \) for \( i = 1, 2, 3, \ldots \) so that

\[
X = \bigcup_{i=1}^{\infty} K_i.
\]

Recall a topological space \( X \) is Hausdorff if for every \( x, y \in X, x \neq y \), there are open neighborhoods \( \mathcal{O}_x \ni x, \mathcal{O}_y \ni y \) so that \( \mathcal{O}_x \cap \mathcal{O}_y = \emptyset \).

A topological manifold of dimension \( k \) is a \( \sigma \)-compact, Hausdorff topological space \( X \) which is locally homeomorphic to \( \mathbb{R}^k \). In other words, around every \( x \in X \), there is a neighborhood \( W \) which is homeomorphic to an open set \( U \subset \mathbb{R}^k \). The homeomorphism \( \phi : U \to W \) is as usual called a parametrization of the neighborhood \( W \) of \( x \). \( \phi^{-1} \) is called a local coordinate system on the neighborhood \( X \) near \( x \).

**Proposition 1.** Any submanifold \( X \) of \( \mathbb{R}^N \) is Hausdorff and \( \sigma \)-compact.

**Proof.** That \( X \) is Hausdorff follows from the fact that \( X \) is a metric space with the induced metric from \( \mathbb{R}^N \).

To prove that \( X \) is \( \sigma \)-compact, define \( bX = \overline{X} \setminus X \). (We can think of \( bX \) as the boundary of the manifold \( X \).) Then if \( bX \neq \emptyset \), let

\[
K_n = \{ x \in X : |x| \leq n, \text{dist}(x, bX) \geq \frac{1}{n} \}.
\]

Here \( \text{dist}(x, bX) = \inf \{|x - y| : y \in bX\} \). In the case \( bX = \emptyset \), then define

\[
K_n = \{ x \in X : |x| \leq n \}.
\]

It is an exercise to show that \( X = \bigcup_{n=1}^{\infty} K_n \), and that each \( K_n \) is closed and bounded in \( \mathbb{R}^N \) (and is thus compact).

2. Smooth manifolds

So far we cannot do any calculus on our manifold \( X \), since it is just a topological space (and not necessarily a subset of \( \mathbb{R}^N \), as we’ve considered before). So we cannot talk about tangent spaces, immersions, transversal intersections, etc., yet.
For starters, let’s consider real valued functions on $X$. So $f : X \to \mathbb{R}$. What should it mean for such a function to be smooth? Certainly we want the corresponding function to be smooth in each coordinate chart $U$. In other words, if $\phi : U \to X$ is a local parametrization, then we want $f \circ \phi : U \to \mathbb{R}$ to be smooth. Note that this makes sense because $U$ is an open subset of $\mathbb{R}^k$.

There is a potential problem, however. A given point $x \in X$ is typically contained in more than one coordinate chart $\phi(U) \subset X$. We must make sure that our notion of smoothness is independent of the coordinate chart we choose. So consider the case when $x \in \phi_1(U_1) \cap \phi_2(U_2)$, where for $\alpha = 1, 2, U_\alpha$ is an open subset of $\mathbb{R}^k$ and $\phi_\alpha : U_\alpha \to X$ is a coordinate parametrization. Then a given $f : X \to \mathbb{R}$ is smooth if each of the $f \circ \phi_\alpha$ is smooth. Note each $\phi_\alpha$ is a homeomorphism onto its image. Now let $W_\alpha = \phi_\alpha(U_\alpha)$. Each $W_\alpha$ is a neighborhood of $x$, and let $W_{12} = W_1 \cap W_2$ be the intersection. Then, on the restricted domains $\phi^{-1}_\alpha(W_{12})$ (i.e. we’re shrinking each $U_\alpha$)

$$\phi_\alpha : \phi^{-1}_\alpha(W_{12}) \to W_{12}$$

is a homeomorphism. Now let’s assume $f \circ \phi_1$ is smooth. Then on restricted domain at least,

$$f \circ \phi_2 = (f \circ \phi_1) \circ (\phi^{-1}_1 \circ \phi_2).$$

Then it is possible to differentiate $f \circ \phi_2$, as long as $\phi^{-1}_1 \circ \phi_2$ is differentiable (by the chain rule). And if we want to compute higher derivatives, we’ll need all the higher derivatives of $\phi^{-1}_1 \circ \phi_2$ as well. The condition we require is that

$$\phi^{-1}_1 \circ \phi_2 : \phi^{-1}_2(W_{12}) \to \phi^{-1}_1(W_{12})$$

be a diffeomorphism. Note that since the domain and range of this function are open subsets of $\mathbb{R}^k$, it makes sense to talk about $\phi^{-1}_1 \circ \phi_2$ being differentiable. Also, our function is already a homeomorphism by the fact $X$ is a topological manifold: what we need in addition is smoothness and smoothness of the inverse.

Before we define smooth manifolds, one more definition is useful: an atlas of a manifold $X$ of dimension $k$ is a collection of parametrizations $\{(U_\alpha, \phi_\alpha)\}$—i.e. $U_\alpha \subset \mathbb{R}^k$ is an open subset and $\phi_\alpha : U_\alpha \to X$ is a homeomorphism onto its image $W_\alpha$—so that

$$X = \bigcup_\alpha \phi_\alpha(U_\alpha).$$

In other words, the coordinate charts $W_\alpha = \phi_\alpha(U_\alpha)$ form an open cover of $X$. 

A smooth manifold $X$ is a topological manifold together with an atlas $(U_\alpha, \phi_\alpha)$ so that each
\[ \phi^{-1}_\alpha \circ \phi_\beta \]
is a diffeomorphism. Each such function $\phi^{-1}_\alpha \circ \phi_\beta$ is called a gluing map. If the two coordinate charts $W_\alpha \cap W_\beta = \emptyset$, then this condition is vacuous. (As above, $W_\alpha = \phi_\alpha(U_\alpha).$) In general, as above, the domain and range of $\phi^{-1}_\alpha \circ \phi_\beta$ are open subsets of $\mathbb{R}^k$. The domain is $\phi^{-1}_\beta(W_{\alpha\beta})$, and the range is $\phi^{-1}_\alpha(W_{\alpha\beta})$, where $W_{\alpha\beta} = W_\alpha \cap W_\beta$. In this case when each gluing map $\phi^{-1}_\alpha \circ \phi_\beta$ is a diffeomorphism, the atlas $(U_\alpha, \phi_\alpha)$ is called a smooth atlas of $X$.

A useful way of thinking of a manifold with an atlas is as a union of the coordinate charts glued, or patched, together according to the functions $\phi^{-1}_\alpha \circ \phi_\beta$. Our manifold $X$ can be identified as a set with the union of the coordinate neighborhoods $U_\alpha$ modulo patching together. More specifically, for our atlas $(U_\alpha, \phi_\alpha)$, consider the disjoint union
\[ U = \bigsqcup_a U_\alpha. \]
Then the patching together is performed by an equivalence relation. Two points $x \in U_\alpha$, $y \in U_\beta$ are equivalent if they map to the same point in our manifold $X$. In terms of the atlas, this just means that
\[ x \sim y \iff x = (\phi^{-1}_\alpha \circ \phi_\beta)(y). \]
Then as a set at least $X$ is equal to the quotient $U/\sim$.

If $X$ and $Y$ are smooth manifolds of respective dimensions $k$ and $\ell$, then a map $f : X \to Y$ is smooth if it is smooth on every coordinate chart. In other words, for all $x \in X$, if $y = f(x)$, then we can find coordinate charts in our smooth atlases so that $f$ is a smooth function in these coordinates. More explicitly, if $x \in \phi(U)$, $y \in \psi(V)$, for smooth parametrizations $\phi$ and $\psi$, then we require
\[ \psi^{-1} \circ f \circ \phi \]
to be smooth. Note the domain of this function is $U \cap \phi^{-1}(\psi(V))$ (so we've had to shrink $U$ a little). This definition of $f$ smooth is independent of the coordinates chosen by the fact that the atlases are smooth.

3. Tangent vectors

For a nice description of tangent vectors and the tangent bundle for manifolds, see the book by Warner, *Foundations of Differentiable Manifolds and Lie Groups*. 

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In particular, the tangent space $T_xX$ of a smooth manifold at a point $x$ can be defined independently of any inclusion of $X \to \mathbb{R}^N$ as a submanifold.

4. AN EXHAUSTION OF ANY MANIFOLD

Let $X$ be a topological space. A countable collection of open subsets $\{O_i\}_{i=1}^{\infty}$ is an exhaustion of $X$ if

- $X = \bigcup_{i=1}^{\infty} O_i$, and
- $\overline{O_i}$ is compact and $\overline{O_i} \subset O_{i+1}$.

(The last pair of conditions is denoted $O_i \subset \subset O_{i+1}$.)

**Proposition 2.** Any manifold $X$ has an exhaustion by open subsets.

**Proof.** Since $X$ is sigma-compact,

$$X = \bigcup_{i=1}^{\infty} K_i, \quad K_i \text{ compact.}$$

An easy lemma we will use repeatedly is this:

**Lemma 3.** A finite union of compact subsets of a topological space is compact.

The proof is left as an exercise.

Let $O_0 = \emptyset$. We prove the Proposition by induction on the following statement for $n > 0$: There are open sets $O_i$ for $i = 1, \ldots, n$ so that

- $O_i \supset K_i \cup \overline{O_{i-1}}.$
- $\overline{O_i}$ is compact.

To complete the initial $n = 1$ step of the induction, we need only construct $O_1$. Define $\tilde{K}_1 = K_1$. For all $x \in \tilde{K}_1$, there is a coordinate neighborhood $W = \phi(U), \ U \subset \mathbb{R}^k$ open. We may assume $\phi(0) = x$. Then let $N_x$ be a neighborhood of $x$, so that $N_x \subset W$. (This is possible since we can always find a small ball $\phi(B_\epsilon(0)) \subset W$. Then take $N_x = \phi(B_{\epsilon/2}(0))$. That $N_x \subset \subset W$ follows from Proposition 4 below.) Now $\{N_x\}_{x \in \tilde{K}_1}$ is an open cover of $\tilde{K}_1$, and thus has a finite subcover $\{N_{x_i}\}_{i=1}^{M}$. We let

$$O_1 = \bigcup_{i=1}^{M} N_{x_i}.$$ 

Then by the lemma above, $\overline{O_1} = \bigcup_{i=1}^{M} \overline{N_{x_i}}$ is compact.

Now assume by induction, we have open sets $O_1, \ldots, O_n$ satisfying the inductive criteria: So in particular, $O_n$ is compact and $O_n \supset$
$K_n \cup \overline{O_{n-1}}$. Now let $\bar{K}_{n+1} = K_{n+1} \cup \overline{O_n}$. By the lemma above and the induction hypothesis, $\bar{K}_{n+1}$ is compact. Now we can apply the argument of the previous paragraph to define

$$\mathcal{O}_{n+1} = \bigcup_{i=1}^{M} N_{x_i},$$

for \(\{N_{x_i}\}_{i=1}^{M}\) a finite open cover of $\bar{K}_{n+1}$ so that each $N_{x_i}$ is compact. It is easy to check the induction step now.

Since each $\mathcal{O}_n \supset K_n$, $X = \bigcup_{n=1}^{\infty} \mathcal{O}_n$. Also, by induction, $\mathcal{O}_n \subset \subset \mathcal{O}_{n+1}$. □

5. The Hausdorff property and manifolds

In this section, we drop temporarily the assumption that manifolds are Hausdorff in order to discuss a useful property which is equivalent to the Hausdorff property for manifolds.

**Proposition 4.** Let $X$ be a manifold of dimension $n$. The following property is equivalent to $X$ being Hausdorff:

For every local parametrization given by $(U, \phi)$—in other words, $U \subset \mathbb{R}^n$ is open and $\phi: U \rightarrow X$ is a homeomorphism onto its image—and for every open $V \subset U$ so that $\overline{V} \subset U$ is compact, then $\overline{\phi(V)} = \phi(\overline{V})$.

**Proof.** First of all, let us assume this property and show $X$ is Hausdorff. Let $x \neq y$ be points in $X$. If $x$ and $y$ are in a single coordinate chart $\phi(U)$, then since $U$ is open, there are small open balls $B_x$ and $B_y$ centered at $\phi^{-1}(x)$ and $\phi^{-1}(y)$ contained in $U \subset \mathbb{R}^n$. Then choose the radii of these balls to be strictly less than $\frac{1}{2} |\phi^{-1}(x) - \phi^{-1}(y)|$ to guarantee they are disjoint (by the triangle inequality). Then $\phi(B_x)$ and $\phi(B_y)$ are the required disjoint open sets in $X$.

If $x \neq y$ are not in a single coordinate chart, then let $x \in \phi(U)$, $y \notin \phi(U)$. Then choose a small open ball $V$ centered at $\phi^{-1}(x)$ in $U \subset \mathbb{R}^n$ so that $\overline{V} \subset U$. Then since $\overline{\phi(V)} = \phi(\overline{V}) \subset \phi(U)$, $y \in X \setminus \phi(V)$, which is an open set disjoint from the open neighborhood $\phi(V)$ of $x$. Therefore $X$ is Hausdorff.

Now assume $X$ is Hausdorff. Let $U \subset \mathbb{R}^n$ be a coordinate chart with parametrization $\phi$ and an open subset $V \subset U$ so that $\overline{V} \subset U$ is compact. We want to show that $\overline{\phi(V)} = \phi(\overline{V})$.

First we show $\overline{\phi(V)} \supset \phi(\overline{V})$. This just uses the fact $\phi$ is continuous: If $W$ is an open subset of $X$ which does not intersect $\phi(V)$, then $\phi^{-1}(W)$ is an open subset of $U$ which does not intersect $V$. Thus $\phi^{-1}(W) \cap \overline{V} = \emptyset$ by the definition of $\overline{V}$. Therefore, $W \cap \phi(V) = \emptyset$. Since $W$ is contained in the complement of $\overline{\phi(V)}$, then $\overline{\phi(V)} \supset \phi(\overline{V})$.  


To show $\phi(V) = \phi(V)$, it suffices to show $\phi(V)$ is closed. Since $V$ is compact and $\phi$ is continuous, $\phi(V)$ is compact. And since $X$ is Hausdorff, $\phi(V)$ is closed. □

Remark. Notice that the Hausdorff property of manifolds then implies that they are locally compact. In other words, every $x \in X$ has a neighborhood $W$ with compact closure.

6. An example

Real projective space $\mathbb{R}P^n$ is an $n$-dimensional smooth manifold which is not be naturally defined as a subset of $\mathbb{R}^N$. Instead, the definition is in terms of the quotient topology. Consider the set $S = \mathbb{R}^{n+1} \setminus \{0\}$. We define an equivalence relation on $S$. If $p, q \in S$, we say $p \sim q$ if there is a nonzero real number $\gamma$ such that $p = \gamma q$. Notice that $p \sim q$ if and only if $p$ and $q$ are on the same line through the origin. Form the quotient space $\mathbb{R}P^n = S/\sim$, and give $\mathbb{R}P^n$ the quotient topology. Thus $\mathbb{R}P^n$ is naturally the set of lines through the origin in $\mathbb{R}^{n+1}$. The quotient projection from $S$ to $\mathbb{R}P^n$ is denoted by $\pi: p \mapsto [p]$.

We claim that $\mathbb{R}P^n$ is naturally a smooth manifold. First we check the topological properties.

Lemma 5. $\mathbb{R}P^n$ is Hausdorff.

Proof. A set $U \subset \mathbb{R}P^n$ is open if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. The set $\pi^{-1}(U)$ is a collection of lines, in other words a cone with the origin deleted. If $[p] \neq [q]$, then the two lines $\pi^{-1}(p)$ and $\pi^{-1}(q)$ are distinct lines through the origin. Notice that they don’t meet in $S$. Let $\theta$ be the angle between these two lines. Then let

$$C_p = \bigcup_{\angle(\ell, \pi^{-1}(p)) < \theta/2} \ell \setminus \{0\}$$

Here $\ell$ represents any line through the origin in $\mathbb{R}^{n+1}$ which meets $\pi^{-1}(p)$ with angle less than $\theta/2$. Similarly define $C_q$. Then since $C_p \cap C_q = \emptyset$, and both are open cones in $S$, the sets $U_p = \pi(C_p)$ and $U_q = \pi(C_q)$ are disjoint neighborhoods of $[p],[q]$. □

Lemma 6. $\mathbb{R}P^n$ is compact (and is therefore sigma-compact).

Proof. $\mathbb{R}P^n = \pi(S^n)$, where $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$, which is compact. Then since $\pi$ is continuous, $\mathbb{R}P^n$ is compact. □

To order to prove $\mathbb{R}P^n$ is a manifold, then we need to find a smooth atlas. There is a standard choice of $n + 1$ coordinate parametrizations which make up an atlas of $\mathbb{R}P^n$. Define for each $i = 1, \ldots, n + 1$,

$$P_i = \{x = (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i = 1\}.$$
It is obvious that each $P_i$ is diffeomorphic to $\mathbb{R}^n$ (the functions $\{x^j\}_{j \neq i}$ are $n$ coordinate functions). If $p = (p^1, \ldots, p^{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$, then at least one coordinate $p^i \neq 0$. Then $[p] = [\frac{1}{p^i} p]$, and $\frac{1}{p^i} \in P_i$. This shows that every $[p] \in \mathbb{RP}^n$ is contained in $\pi(P_i)$ for at least one $P_i$. In other words,

$$\mathbb{RP}^n = \bigcup_{i=1}^{n+1} \pi(P_i)$$

This is an open cover of $\mathbb{RP}^n$ since each

$$\pi^{-1}(\pi(P_i)) = \{x \in \mathbb{R}^{n+1} : x^i \neq 0\}$$

is open. Let $U_i = \pi(P_i)$. Note that $U_i = \{[p] : p^i \neq 0\}$. It is straightforward to check that $\pi : P_i \to U_i$ is a homeomorphism.

Now we must check that the for these coordinates, the gluing maps are diffeomorphisms. For each $i \in \{1, \ldots, n+1\}$, let

$$x_i = (x^1_i, \ldots, x^i_{-1}, x^{i+1}_i, \ldots, x^n_i)$$

be coordinates in $\mathbb{R}^n$. Also define a diffeomorphism $\chi_i : \mathbb{R}^n \to P_i$ by

$$\chi_i(x_i) = (x^1_i, \ldots, x^{i-1}_i, 1, x^{i+1}_i, \ldots, x^{n+1}_i).$$

Then let $\phi_i = \pi \circ \chi_i : \mathbb{R}^n \to U_i$. These are the coordinate parametrizations. We need to check that on $\phi_i^{-1}(U_i \cap U_k), \phi_k^{-1} \circ \phi_i$ is smooth. Now for $[p] \in U_i \cap U_k, p^i \neq 0, p^k \neq 0$. Therefore,

$$\phi_i(x_i) \in U_i \cap U_k \iff x^k_i \neq 0.$$

Compute

$$\phi_k^{-1}(\phi_i(x_i)) = \phi_k^{-1}(\begin{bmatrix} x^1_i \\ \vdots \\ x^{i-1}_i \\ 1 \\ x^{i+1}_i \\ \vdots \\ x^{n+1}_i \end{bmatrix}) = \phi_k^{-1}(\begin{bmatrix} x^1_i \\ \vdots \\ x^{i-1}_i \\ 1 \\ x^{i+1}_i \\ \vdots \\ x^{n+1}_i \end{bmatrix}) = (\begin{bmatrix} x^1_i \\ \vdots \\ x^{i-1}_i \\ 1 \\ x^{i+1}_i \\ \vdots \\ x^{n+1}_i \end{bmatrix}).$$

Since $x^k_i \neq 0$, then it’s clear this function is smooth. The inverse is smooth since $(\phi_k^{-1} \circ \phi_i)^{-1} = \phi_i^{-1} \circ \phi_k$ and we may repeat the argument with $i$ and $k$ interchanged.