PARTITIONS OF UNITY

Let $f : X \to \mathbb{R}$ be a function from a topological space X. The *support* of f is the closed set

$$\operatorname{supp} f = \overline{f^{-1}(\mathbb{R} \setminus \{0\})}.$$

So the complement of the support is the largest open set on which f is zero. Given a collection of functions $f_\alpha \colon X \to \mathbb{R}$, the sum

$$\sum_\alpha f_\alpha$$

is *locally finite* if for every point $x \in X$, there is a neighborhood on which all but finitely many of the f_α are identically zero. In other words, this neighborhood intersects the supports of only finitely many f_α. Note that if X is a smooth manifold and all of the functions f_α are smooth, then the locally finite sum $\sum_\alpha f_\alpha$ is also a smooth function since near each point, we only have to take the derivative of a finite sum.

Let $\{W_\alpha\}$ be an open cover of a smooth manifold X. Then a *partition of unity* subordinate to the open cover is a collection of smooth functions $\{\theta_i\}$ so that

1. $0 \leq \theta_i(x) \leq 1$ for all $x \in X$, and for all i.
2. Each $\operatorname{supp} \theta_i \subset\subset W_\alpha$ for some α.
3. The sum $\sum_i \theta_i$ is locally finite.
4. $\sum_i \theta_i(x) = 1$. The previous conditions guarantees this sum to be finite for each $x \in X$.

Theorem 1. *Let $\{W_\alpha\}$ be an open cover of a smooth manifold X. Then there exists a partition of unity subordinate to this open cover.*

Proof. Let $\mathcal{O}_i$ be an exhaustion of X by open subsets (Proposition 2 in "The Real Definition of a Smooth Manifold"). In other words $X = \bigcup_{i=1}^\infty \mathcal{O}_i$ and $\mathcal{O}_i \subset\subset \mathcal{O}_{i+1}$. Then $\overline{\mathcal{O}_i} \setminus \mathcal{O}_{i-1}$ is compact. Then there is an open cover of $\overline{\mathcal{O}_i} \setminus \mathcal{O}_{i-1}$ consisting of sets $\phi(B)$ so that

1. Each $\phi : U \to X$ is a local parametrization map and each B is an open ball in $\mathbb{R}^n$ so that $B \subset\subset U$. (So then each $\phi(B)$ is open.)
2. Each $\phi(B) \subset\subset W_\alpha$ for some α. We can do this because sets of the form $\phi(B)$ form a basis of the topology of X (exercise).
3. Each $\phi(B) \subset\subset \mathcal{O}_{i+1} \setminus \overline{\mathcal{O}_{i-2}}$.

Note that in this case, each $\phi(\overline{B}) = \overline{\phi(B)}$ and $\overline{\phi(B)}$ is compact. We can do this by the Hausdorff property (Proposition 4 in "The Real Definition of a Smooth Manifold"), and property 1 above.

We can find such an open cover because sets of the form $\phi(B)$ form a basis of the topology of X. Extract a finite subcover indexed by j. For each of these sets $\phi_j(B_j)$, we'll have a smooth function $\eta_j\colon X \to \mathbb{R}$. For each $i = 1, \dots, \infty$, we'll define one function η_j for each open set $\phi_j(B_j)$ in the finite subcover. Notice we end up with a countable collection of functions η_j. (This means our finite subcover is indexed by $j \in \{J+1, \dots, J+N\}$, where $j \leq J$ correspond to the steps $1, \dots, i-1$.)

For each ball B, let $\rho_j(x)\colon \mathbb{R}^k \to \mathbb{R}$ so that ρ_j is smooth, supp $\rho_j = \overline{B_j}$, and $\rho_j(y) > 0$ for $y \in B_j$. This is the bump function constructed in "Bump Functions." Assume by induction we've defined $\eta_1, \dots \eta_{j-1}$. For our $\phi_j(B_j)$, let $\phi_j(U_j) = W_\alpha$—and so $B_j \subset\subset U_j$ by property (2). Then define

$$\eta_j(x) = \begin{cases} \rho_j(\phi_j^{-1}(x)) & \text{for } x \in \phi_j(U_j) \\ 0 & \text{for } x \notin \overline{\phi_j(B_j)}. \end{cases}$$

Thus supp $\eta_j = \overline{\phi_j(B_j)}$. This definition is well defined: First notice that $\overline{\phi_j(B_j)} = \phi_j(\overline{B_j}) \subset \phi_j(U_j)$, which is true by Proposition 4 in "The Real Definition of a Smooth Manifold." Thus the two sets in the definition cover X. Also, note that $\rho_j(y) = 0$ if and only if $y \in B_j$, and so the two definitions of η_j agree on the overlap $\phi(U_j) \setminus \overline{\phi(B_j)}$.

To check that η_j is smooth, ϕ_j^{-1} and ρ_j are smooth. Since the two sets $\phi_j(U_j)$ and $X \setminus \overline{\phi_j(B_j)}$ are open, and the definitions are smooth and agree on the overlap, then η_j must be smooth.

By property (3) above, the sum

$$\sum_{j=1}^{\infty} \eta_j$$

is locally finite. The sum is always positive since $\{\phi_j(B_j)\}$ is an open cover of X, and $\eta_j > 0$ on $\phi_j(B_j)$. Property (2) above guarantees that supp $\eta_j \subset$ some W_α. Now define

$$\theta_j = \frac{\eta_j}{\displaystyle\sum_{\ell=1}^{\infty} \eta_\ell}.$$

It's easy to verify that θ_j is a partition of unity subordinate to $\{W_\alpha\}$. $\square$

To show the usefulness of this proposition, consider the following extension lemma:

Lemma 1. *Let* X *be a smooth manifold and* $C \subset V \subset X$*, where* C *is closed and* V *is an open neighborhood of* C*. Let* $f : V \to \mathbb{R}$ *be a smooth function. Then there is a smooth function* $F : X \to \mathbb{R}$ *so that* $F(x) = f(x)$ *for* $x \in C$ *and supp* $F \subset V$*. In other words* F *extends* f *on* C *and the support of* F *is contained in the neighborhood* V *of* C*.*

Proof. $\{V, X \setminus C\}$ is an open cover of X. Find a partition of unity θ_i subordinate to this open cover. Then let

$$\sigma = \sum_{\mathrm{supp}\,\theta_{\mathrm{i}} \subset \mathrm{V}} \theta_i. \tag{1}$$

Then $\sigma = 1$ on C, since $\sum_i \theta_i = 1$ and since all the terms left out of the sum (1) have support contained in $X \setminus C$.

Claim. *Supp* $\sigma \subset V$*.*

Proof. Consider the set

$$S = \bigcup_{\mathrm{supp}\,\theta_{\mathrm{i}} \subset \mathrm{V}} \mathrm{supp}\ \theta_i.$$

We will show S is closed and $S =$ supp σ. Note $S \subset V$, so this will prove the claim. Consider a point $p \notin S$. Then all θ_i in the sum (1) vanish at p. Moreover, the local finiteness shows that there is an open neighborhood $\mathcal{O} \ni p$, so that the supports of finitely many θ_i intersect $\mathcal{O}$. Then $\mathcal{O}$ minus the supports of these finitely many θ_i is an open neighborhood of p which doesn't intersect S. Thus S is closed. It is easy to see then that $S =$ supp σ and thus supp $\sigma \subset V$. ☐

It is straightforward to check that

$$F(x) = \begin{cases} \sigma(x) f(x) & \text{for } x \in V \\ 0 & \text{for } x \notin V \end{cases}$$

is the desired extension. ☐