PATCHING TOGETHER FUNCTIONS ON MANIFOLDS

Suppose we have a n-dimensional manifold $X \subset \mathbb{R}^N$. We want to define a function $f: X \to Y$, where Y is another manifold. X can be covered by a number of parametrizations. In other words

$$X = \bigcup_i \phi_i(U_i)$$

for $U_i \subset \mathbb{R}^n$ open, and $\phi_i : U_i \to \phi(U_i)$ diffeomorphisms.

One way of defining a function from X to Y is to define maps

$$f_i : U_i \to Y$$

from each coordinate chart to Y. Then we would like these maps to patch together to form a map from X to Y. On $\phi_i(U_i) \subset X$, then we have the natural maps

$$\tilde{f}_i : \phi_i(U_i) \to Y, \qquad \tilde{f}_i(x) = f_i(\phi_i^{-1}(x)).$$

In order for these maps $\tilde{f}_i$ to define a single map f from X to Y, then we must have f single-valued. In other words, if $x \in \phi_i(U_i) \cap \phi_j(U_j)$, then we can define

$$f(x) = \tilde{f}_i(x) = \tilde{f}_j(x)$$

as long as these last two are equal. In terms of the maps f_i on the coordinate charts, we require that

$$f_i \circ \phi_i^{-1} = f_j \circ \phi_j^{-1} \quad \text{on} \quad \phi_i(U_i) \cap \phi_j(U_j).$$

So we must be able to patch f together on the intersection of any two coordinate charts.

Example 1. *Recall that* $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ *can be covered by two coordinate charts*

$$\begin{aligned} \theta_1 \in (0, 2\pi), &\quad \phi_1(\theta_1) = (\cos\theta_1, \sin\theta_1); \\ \theta_2 \in (-\pi, \pi), &\quad \phi_2(\theta_2) = (\cos\theta_2, \sin\theta_2). \end{aligned}$$

We want to define a map from S^1 *to* S^1 *by* $\theta \mapsto 2\theta$*. More precisely, on our two coordinate charts, we define*

$$f_1(\theta_1) = (\cos 2\theta_1, \sin 2\theta_1), \qquad f_2(\theta_2) = (\cos 2\theta_2, \sin 2\theta_2).$$

Do these patch together to make a map $f : S^1 \to S^1$*?*

Our two coordinate charts overlap in two pieces: $S^1 \cap \{y > 0\}$ *and* $S^1 \cap \{y < 0\}$. *On* $S^1 \cap \{y > 0\}$, *we're in the first and second quadrants in the plane, and* $\theta_1, \theta_2 \in (0, \pi)$. *In fact* $\theta_1 = \theta_2$. *In this case,*

$$f_1(\theta_1) = f_2(\theta_2) = (\cos 2\theta_1, \sin 2\theta_1).$$

On the other component of the intersection $S^1 \cap \{y < 0\}$, *in the third and fourth quadrants, we have* $\theta_1 \in (\pi, 2\pi)$, *but* $\theta_2 \in (-\pi, 0)$, *and* $\theta_1 = \theta_2 + 2\pi$. *Then*

$$\begin{aligned} f_1(\theta_1) &= (\cos 2\theta_1, \sin 2\theta_1) \\ &= (\cos(2\theta_2 + 4\pi), \sin(2\theta_2 + 4\pi)) \\ &= (\cos 2\theta_2, \sin 2\theta_2) \\ &= f_2(\theta_2). \end{aligned}$$

Thus these 2 maps on coordinates patch together to form a map from S^1 *to* S^1.

Remark. The smoothness of the map f in the above example is obvious by the description in coordinate charts. It is clear for example that $f_1 : \theta_1 \mapsto (\cos 2\theta_1, \sin 2\theta_1)$ is smooth. Therefore, since ϕ_1^{-1} is smooth (by the definition of a parametrization), we see that $f = f_1 \circ \phi_1^{-1}$ is smooth on the domain of ϕ_1^{-1}. (The composition of 2 smooth maps is smooth by the chain rule.) Similarly, we can show that f is smooth on the domain of ϕ_2^{-1} and thus on all of S^1.

Example 2. *With the same setup as above, the map on coordinates* $\theta \mapsto \frac{3}{2}\theta$ *does not patch. Similarly to above, define*

$$f_1(\theta_1) = (\cos \tfrac{3}{2}\theta_1, \sin \tfrac{3}{2}\theta_1), \qquad f_2(\theta_2) = (\cos \tfrac{3}{2}\theta_2, \sin \tfrac{3}{2}\theta_2).$$

The key computation is the last one, in which we find on $S^1 \cap \{y < 0\}$,

$$\begin{aligned} f_1(\theta_1) &= (\cos \tfrac{3}{2}\theta_1, \sin \tfrac{3}{2}\theta_1) \\ &= (\cos(\tfrac{3}{2}\theta_2 + 3\pi), \sin(\tfrac{3}{2}\theta_2 + 3\pi)) \\ &= (-\cos \tfrac{3}{2}\theta_2, -\sin \tfrac{3}{2}\theta_2) \\ &\neq f_2(\theta_2). \end{aligned}$$

Thus in this case, the maps do not patch to form a map from S^1 *to* S^1.

Note that in these examples, for the source S^1, we have used not the representation of S^1 as a submanifold of $\mathbb{R}^2$, but the coordinates θ_1 and θ_2 and how they relate to each other on the overlap. This idea will help us when we define manifolds in the abstract (i.e. not merely as subsets of $\mathbb{R}^N$).