AN IMMERSION WITH DENSE IMAGE

Recall that $S^1 \times S^1 \subset \mathbb{R}^4$ can be expressed as the image of the immersion

$$f\colon \mathbb{R}^2 \to \mathbb{R}^4, \qquad f(\alpha_1, \alpha_2) = (\cos 2\pi\alpha_1, \sin 2\pi\alpha_1, \cos 2\pi\alpha_2, \sin 2\pi\alpha_2).$$

(Note this is not quite the same as our earlier description; typically we would instead use $\theta_i = 2\pi\alpha_i$ for $i = 1, 2$.) Then consider a line through the origin in $\mathbb{R}^2$ given by the parametrization $g(t) = (t, \gamma t)$. We assume γ is an *irrational* number.

Theorem 1. *Let $h\colon \mathbb{R} \to S^1 \times S^1$ be given by $h(t) = f(g(t))$, for f and g defined above. Then $h(t)$ is a injective and the image $h(\mathbb{R})$ is dense in $S^1 \times S^1$.*

Remark. Recall that for a topological space Y, $X \subset Y$ is *dense* if $\overline{X} = Y$. This is equivalent to the following condition: For each nonempty open set $\mathcal{O} \subset Y$, then $\mathcal{O} \cap X \neq \emptyset$.

First of all, it is useful to have a description of $S^1 \times S^1$ in terms of α_1, α_2. The immersion f is periodic with period 1 in α_1 and α_2 (since sin and cos are periodic with period 2π). In other words, if $k, \ell \in \mathbb{Z}$, then

$$f(\alpha_1 + k, \alpha_2 + \ell) = f(\alpha_1, \alpha_2).$$

If we restrict $(\alpha_1, \alpha_2) \in [0, 1) \times [0, 1)$, then

$$f\colon [0, 1) \times [0, 1) \to S^1 \times S^1$$

is one-to-one and onto. (Note this is *not* a parametrization of $S^1 \times S^1$, since the domain $[0, 1) \times [0, 1)$ is not open. Moreover, the inverse map is not continuous. Why?) If $\alpha_1, \alpha_2 \in \mathbb{R}$, then

$$f(\alpha_1, \alpha_2) = f(\alpha_1 - [\alpha_1], \alpha_2 - [\alpha_2])$$

Here $[x]$ is the largest integer $\leq x$, and thus for any $x \in \mathbb{R}$, $x - [x] \in [0, 1)$ is the fractional part of x.

Thus if we want to work in the half-open square $[0, 1) \times [0, 1)$, then we can use the fractional part of α_1, α_2 instead. In particular, we can write

$$h = f \circ g = f \circ \tilde{g}, \qquad \tilde{g}(t) = (t - [t], \gamma t - [\gamma t]) \tag{1}$$

Note the image of $\tilde{g}$ is inside $[0, 1) \times [0, 1)$, on which f is one-to-one. g parametrizes the line through the origin in $\mathbb{R}^2$ with irrational slope γ.

Proposition 1. *h is one-to-one.*

Proof. Since f is one-to-one on $[0,1) \times [0,1)$, equation (1) shows us that it suffices to check that $\tilde{g}$ is one-to-one.

So if $\tilde{g}(t) = \tilde{g}(s)$, then

$$t - [t] = s - [s], \qquad \gamma t - [\gamma t] = \gamma s - [\gamma s].$$

The first equality implies that $t - s = [t] - [s] \in \mathbb{Z}$. Then if $t \neq s$, we can solve the second equation to find

$$\gamma = \frac{[\gamma t] - [\gamma s]}{t - s} \in \mathbb{Q}.$$

This contradicts our assumption that γ is irrational. So therefore if $\tilde{g}(t) = \tilde{g}(s)$, then $t = s$. So $\tilde{g}$ is one-to-one. □

If $t = n$, $s = m$ are integers, then we have the corollary of the proof

Corollary 2. *If $n \neq m$ are integers, then $\gamma n - [\gamma n] \neq \gamma m - [\gamma m]$.*

Now we'll proceed to prove that $\tilde{g}(\mathbb{R})$ is dense in $[0,1) \times [0,1)$. The following proposition starts us on the road to proving the related fact that $\tilde{g}(\mathbb{Z})$ is dense in $\{0\} \times [0,1)$.

Proposition 3. *For any integer i, define $x_i = \gamma i - [\gamma i]$. Then for any positive integer q, there is a nonzero integer n so that $x_n \in (0, 1/q)$.*

Proof. First of all, we have the following disjoint union of $[0,1)$:

$$[0,1) = \bigcup_{j=0}^{q-1} \left[\frac{j}{q}, \frac{j+1}{q}\right).$$

Then by Corollary 2, the first $q+1$ terms $x_1, x_2, \ldots, x_{q+1}$ are distinct from each other. Then the pigeonhole principle shows that there is a $j \in \{0, \ldots, q-1\}$ so that for $k \neq \ell \in 1, 2, \ldots, q+1$,

$$x_k, x_\ell \in \left[\frac{j}{q}, \frac{j+1}{q}\right).$$

$x_k \neq x_\ell$ by Corollary 2. Assume without loss of generality that $x_k > x_\ell$. So then $x_k - x_\ell \in (0, 1/q)$. Compute

$$x_k - x_\ell = \gamma k - [\gamma k] - \gamma \ell + [\gamma \ell] = \gamma(k - \ell) - [\gamma(k - \ell)] = x_{k-\ell}.$$

The second equality follows from Lemma 4 below. It is a consequence of the fact that $x_k - x_\ell \in [0,1)$. So we have shown that $x_{k-\ell} \in (0, 1/q)$. □

Lemma 4. *Define $\phi(x) = x - [x]$. Then for any $x, y \in \mathbb{R}$, $\phi(x+y) = \phi(\phi(x) + \phi(y))$.*

Proof. Exercise. □

Remark. The previous lemma is familiar from modular arithmetic. In fact, in working with the fraction part of real numbers, we're doing arithmetic of real numbers modulo 1.

Proposition 5. *For any positive integer q and $p \in \{0, 1, \ldots, q-1\}$, there is an integer m so that*

$$x_m \in \left(\frac{p}{q}, \frac{p+1}{q}\right).$$

Proof. Let $x_n \in (0, 1/q)$ by Proposition 3 above. Then we'll choose a positive-integral multiple of x_n that is in the specified interval. First of all, by induction and Lemma 4, we find that if a is a positive integer, then $\phi(ax_n) = x_{an}$. Then it is easy to show that

$$a = \left[\frac{p}{qx_n}\right] + 1$$

forces ax_n to be in the specified interval. (The proof also uses the fact that $x_n \notin \mathbb{Q}$ for $n \neq 0$.) □

Proposition 6. $\{x_n\}$ *is dense in* $[0, 1)$.

Proof. We need to show that for any nonempty open set $\mathcal{O} \subset [0, 1)$, there is an $m \in \mathbb{Z}$ so that $x_m \in \mathcal{O}$. Any such $\mathcal{O}$ must contain an interval of the form $(\frac{p}{q}, \frac{p+1}{q})$ as in Proposition 5 above. (This can be proved for q a power of 10 by considering decimal expansions of the endpoints of an interval contained in $\mathcal{O}$.) □

Corollary 7. $\tilde{g}(\mathbb{Z})$ *is dense in* $\{0\} \times [0, 1)$.

To prove the theorem, it suffices to prove that $\tilde{g}(\mathbb{R}^2)$ is dense in $[0, 1) \times [0, 1)$. This is because for

$$(0, 1) \times (0, 1) \subset [0, 1) \times [0, 1),$$

$f((0, 1) \times (0, 1))$ is dense in $S^1 \times S^1$ (Exercise). This f is a coordinate parametrization of $S^1 \times S^1$.

First of all, any nonempty open subset of $[0, 1) \times [0, 1)$ contains an open ball $B_\epsilon(\alpha_1, \alpha_2)$ for $(\alpha_1, \alpha_2) \in (0, 1) \times (0, 1)$ and $\epsilon > 0$. (This requires a proof, because open sets of $[0, 1) \times [0, 1)$ are with respect to the subspace topology induced from $\mathbb{R}^2$. Prove this as an exercise.)

The image of $\tilde{g}$ is a collection of line segments of slope γ. The line of slope γ through (α_1, α_2) has y-intercept $\alpha_2 - \gamma\alpha_1$. Moreover, it is easy to see that for any $\beta \in (\alpha_2 - \gamma\alpha_1 - \epsilon, \alpha_2 - \gamma\alpha_1 + \epsilon)$, the line of slope γ and y-intercept β intersects the line segment $\{\alpha_1\} \times (\alpha_2 - \epsilon, \alpha_2 + \epsilon)$. By taking fractional parts $\phi(x) = x - [x]$, we see that if $\beta \in (\phi(\alpha_2 - \gamma\alpha_1 - \epsilon), \phi(\alpha_2 - \gamma\alpha_1 + \epsilon))$ and $(\beta, 0) = \tilde{g}(n)$, then

$\tilde{g}(n+\alpha_1) = (\alpha_1, \phi(n+\alpha_1)) \in B_\epsilon(\alpha_1, \alpha_2)$. But by Corollary 7, we can always find such a $\beta = \tilde{g}(n)$. Therefore, we have proved the theorem.

Remark. In the above proof, we have implicitly assumed that

$$\phi(\alpha_2 - \gamma\alpha_1 + \epsilon) > \phi(\alpha_2 - \gamma\alpha_1 - \epsilon),$$

which is not true for all choices of $\alpha_1, \alpha_2, \epsilon$. Exercise: Modify the proof to handle the other cases.