

Homework, Differentiable Manifolds, Fall, 2009.

- (1) Let M and N be topological manifolds of dimension m, n respectively. Show that $M \times N$, when equipped with the product topology, is a topological manifold of dimension $m + n$.
- (2) Let M be a topological manifold. Show that M is connected if and only if M is path-connected.
- (3) Consider $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ the standard unit sphere. Let I be the $(n + 1) \times (n + 1)$ identity matrix. Show that the group $\{\pm I\}$ acts freely and properly discontinuously on \mathbb{S}^n .
- (4) Consider $\mathbb{S}^2 \subset \mathbb{R}^3$ the standard unit sphere. Show that the group generated by reflections across the three coordinate planes in \mathbb{R}^3 acts properly discontinuously, but not freely, on \mathbb{S}^2 .
- (5) Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = x^3 - y^2$. For which values $c \in \mathbb{R}$ is the level set $f^{-1}(c)$ a C^∞ submanifold in \mathbb{R}^2 ? Prove your result, and sketch representative level sets.
- (6) Let $f: M \rightarrow N$ be a C^∞ immersion between two connected n -dimensional C^∞ manifolds. Show that if f is proper, then it must be a covering map. (Recall that f is proper if for every compact subset $K \subset N$, $f^{-1}(K)$ is a compact subset of M .)
Hints:
 - (a) Let $p \in N$. Show that each $q \in f^{-1}(p)$ has a neighborhood \mathcal{U}_q diffeomorphic under f to a neighborhood \mathcal{O}_q of p .
 - (b) Show that $f^{-1}(p)$ is a finite set and that $\tilde{\mathcal{O}} = \bigcap_{q \in f^{-1}(p)} \mathcal{O}_q$ is an open neighborhood of p .
 - (c) Shrink $\tilde{\mathcal{O}}$ appropriately so its preimage under f is composed of covering neighborhoods of each $q \in f^{-1}(p)$. The main point is to ensure that $f^{-1}(\tilde{\mathcal{O}})$ is contained in the union of the \mathcal{U}_q . (Hint: for K a compact subset of M , show that $f(f^{-1}(K) \setminus (\bigcup_q \mathcal{U}_q))$ is a compact subset of M which does not contain p . Apply this to a compact set containing all the \mathcal{O}_q .)
- (7) Let $\mathcal{M}_n(\mathbb{R})$ denote the set of $n \times n$ matrices with real coefficients. Note we may identify $\mathcal{M}_n(\mathbb{R})$ with \mathbb{R}^{n^2} . Let $SL(n, \mathbb{R})$ be the set of $n \times n$ real matrices with determinant 1.
 - (a) Show that $SL(n, \mathbb{R})$ is a C^∞ submanifold of $\mathcal{M}_n(\mathbb{R})$. Hint: You may use the following description of the derivative of the determinant function. If $A(t)$ is a differentiable $n \times n$ matrix-valued function with matrix coefficients $A_{ij} =$

$A_{ij}(t)$, then

$$\frac{d}{dt} \det A(t) = (\det A) \sum_{i,j=1}^n A^{ij} \frac{d}{dt} A_{ij},$$

for A^{ij} the inverse matrix of A . Note this formula, as it is written, only makes sense for nonsingular A , but we may rewrite $(\det A)A^{ij}$ as the matrix of cofactors of A , which is well-defined for any matrix A .

- (b) Show that $SL(n, \mathbb{R})$ is a Lie group.
 - (c) Compute the tangent space of $SL(n, \mathbb{R})$ at the identity matrix $I \in \mathcal{M}_n(\mathbb{R})$.
- (8) Consider $\mathbb{R}\mathbb{P}^n$ as the set of all lines through the origin in \mathbb{R}^{n+1} . Let

$$\tau = \{(x, \ell) \in \mathbb{R}^{n+1} \times \mathbb{R}\mathbb{P}^n : x \in \ell\}.$$

Consider the projection $\pi: \tau \rightarrow \mathbb{R}\mathbb{P}^n$ given by $(x, \ell) \mapsto \ell$.

- (a) Show that $\pi: \tau \rightarrow \mathbb{R}\mathbb{P}^n$ is a C^∞ vector bundle of rank 1 over $\mathbb{R}\mathbb{P}^n$. (τ is called the *tautological bundle*.)
 - (b) For $n = 1$, show that $\mathbb{R}\mathbb{P}^1$ is diffeomorphic to the circle \mathbb{S}^1 , and that the total space of τ over $\mathbb{R}\mathbb{P}^1$ is vector-bundle isomorphic to the Möbius band.
- (9) A vector bundle $\pi: V \rightarrow M$ of rank r over a manifold M is said to be trivial if there is a vector bundle morphism $\theta: M \times \mathbb{R}^r \rightarrow V$ which covers the identity (i.e., it satisfies $\pi \circ \theta(p, 0) = p$), so that θ^{-1} is also a vector bundle morphism.
- (a) A *line bundle* is a vector bundle of rank 1. Show that a line bundle is trivial if and only if it has a C^∞ section s which never vanishes.
 - (b) Show that $T\mathbb{S}^1$ is a trivial line bundle.
- (10) (a) Construct a diffeomorphism between the open unit ball B_1 in \mathbb{R}^n and \mathbb{R}^n itself.
- (b) Find, with proof, an explicit incomplete vector field on \mathbb{R}^n . (Hint: Find an incomplete vector field on B_1 and then push it forward via the diffeomorphism to \mathbb{R}^n .)
- (11) Consider \mathbb{S}^n with the standard metric induced from \mathbb{R}^{n+1} , and let g the metric induced on $\mathbb{R}\mathbb{P}^n$ induced from the covering map $\pi: \mathbb{S}^n \rightarrow \mathbb{R}\mathbb{P}^n$ from the quotient construction $\mathbb{R}\mathbb{P}^n \sim \mathbb{S}^n / \{\pm I\}$. Also consider the inhomogeneous coordinate chart $\mathcal{U}_0 = \pi(\mathbb{R}^{n+1} \setminus \{x^0 = 0\})$ for $\pi: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}\mathbb{P}^n$ the standard projection. Recall the coordinates $\{y_0^1, \dots, y_0^n\}$ on \mathcal{U}_0 given by $\phi([(x^0, \dots, x^n)]) =$

$\left(1, \frac{x^1}{x^0}, \dots, \frac{x^n}{x^0}\right)$ and $\phi^{-1}(1, y_0^1, \dots, y_0^n) = [(1, y_0^1, \dots, y_0^n)]$. Compute the metric g restricted to \mathcal{U}_0 in the $\{y_0^i\}$ coordinates.

Also write down all the geodesics of g in these coordinates (push them forward from \mathbb{S}^n).

- (12) Let \mathcal{O} be a connected domain in \mathbb{R}^n equipped with a Riemannian metric g . Let $p, q \in \mathcal{O}$ and consider the class \mathcal{C} of smooth paths from $[0, 1]$ to \mathcal{O} from p to q . Show that a path γ is a geodesic (i.e., is a critical point for the energy functional E) if and only if γ is a constant-speed critical point of the length functional L . (Note: You need to compute the Euler-Lagrange equations of L ; the local results concerning minimizers we discussed in class cannot immediately be applied to the more general case of critical points.)
- (13) Let $f: (a, b) \rightarrow \mathbb{R}$ be a positive smooth function. Consider the surface of revolution of the graph of f around the x axis, as parametrized by

$$(x, \theta) \mapsto (x, f(x) \cos \theta, f(x) \sin \theta)$$

- (a) In the (x, θ) coordinates, compute the Riemannian metric induced from the standard metric on \mathbb{R}^3 .
- (b) Compute the geodesic equations in these coordinates.
- (c) Show that a circle parametrized for fixed x ,

$$\theta \mapsto (x, f(x) \cos \theta, f(x) \sin \theta)$$

is a geodesic if and only if x is a critical point of f .

- (14) Consider the orthogonal group $O(n) = \{A \in \mathcal{M}_n(\mathbb{R}) : A^\top A = I\}$.
- (a) Compute the tangent space to the identity of $O(n)$.
- (b) Show that for $A, B \in T_I O(n)$, the trace form given by $(A, B) \mapsto \text{tr}(AB)$ is negative definite. (Note the tangent space at the identity of the Lie group $T_I O(n)$ is called the *Lie algebra* associated to the Lie group. In this case, it is customary to denote $T_I O(n)$ by $\mathfrak{o}(n)$.)
- (c) For $k \in O(n)$, define the map $L_k : O(n) \rightarrow O(n)$ by $L_k h = kh$. At $k \in O(n)$, for $V, W \in T_k O(n)$, define $g(V, W) = -\text{tr}(L_{k^{-1}*} V)(L_{k^{-1}*} W)$. Show g defines a Riemannian metric on $O(n)$.
- (d) By construction $L_k^* g = g$ (we say g is left-invariant in this case). Define $R_k : O(n) \rightarrow O(n)$ by $R_k h = hk$. Show that for all $k \in O(n)$, $R_k^* g = g$ also.
- (15) Consider the Poincaré ball model of hyperbolic space, defined below. Recall the hyperboloid model of hyperbolic space is

given by

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : \lambda(x, x) = -1, x^0 > 0\}$$

for $\lambda(x, y) = -x^0y^0 + x^1y^1 + \cdots + x^ny^n$. The metric g on \mathbb{H}^n is defined as the pullback $\iota^*\lambda$ for $\iota: \mathbb{H}^n \rightarrow \mathbb{R}^{n+1}$ the injection map. Consider the map f from \mathbb{H}^n to $B_1(0)$ the unit ball in \mathbb{R}^n , defined by $(0, f(x))$ is the intersection point of the hyperplane $\{x^0 = 0\}$ with the line segment from $x \in \mathbb{H}^n$ to the point $S = (-1, 0, \dots, 0) \in \mathbb{R}^{n+1}$.

- (a) Show f is a diffeomorphism.
- (b) Compute the metric $(f^{-1})^*g$ on $B_1(0)$. Show that it is conformal to the standard flat metric δ_{ij} on $\mathbb{R}^n \supset B_1(0)$ (two Riemannian metrics are conformal if one is equal to the other multiplied by a smooth positive function).
- (c) Compute the hyperbolic distance between any two points in $B_1(0)$. (Feel free to use the results we proved in class about the distance in the hyperboloid model of \mathbb{H}^n .)
- (d) Show that the hyperbolic geodesics in $B_1(0)$ are given, as sets, as circular arcs and line segments in $B_1(0) \subset \mathbb{R}^n$ which are perpendicular (with respect to the flat metric on \mathbb{R}^n) to the boundary $\partial B_1(0)$.

Hint: It helps to normalize the geodesic in time first. In particular, minimize $d((1, 0), \gamma(t))$ in t for $\gamma(t) = x \cosh t + v \sinh t$ as usual. Show that for the new value of t , we may assume $v^0 = 0$.

- (e) If γ_1 and γ_2 are two disjoint complete geodesics in the Poincaré ball model of hyperbolic space, show that the hyperbolic distance between γ_1 and γ_2 is 0 if and only if they share a common limit point on the boundary $\partial B_1(0)$. (The hyperbolic distance between γ_1 and γ_2 is given by $\inf_{s,t} d(\gamma_1(s), \gamma_2(t))$, where d is the distance defined by the hyperbolic metric.)

Hint: To prove \Leftarrow , use 15b and 15d (in particular, the fact that any two geodesics with a common boundary point are tangent at the boundary). To prove \Rightarrow , use the explicit formula in 15c to show that if two geodesics have distinct limit points, these limit points are infinitely far away from each other in hyperbolic space.

- (16) A two-form ω on a real vector space V is said to be *nondegenerate* if the map from $V \rightarrow V^*$ given by $v \mapsto \omega(v, \cdot)$ is a linear isomorphism.

- (a) Show that if there is a nondegenerate two-form ω on a finite dimensional real vector space V , then the dimension of V is even.
- (b) Show that if ω is two-form on V a vector space of dimension $2n$, then the following are equivalent:
- ω is nondegenerate.
 - $\omega^n \neq 0$, where ω^n represents $\omega \wedge \omega \wedge \cdots \wedge \omega$ the wedge product of n copies of ω .
- (17) (a) On \mathbb{R}^2 , show that the line integral of the one-form $\omega = y dx + dy$ depends on the choice of path from $(0, 0)$ to $(1, 1)$.
- (b) On $\mathbb{R}^2 \setminus \{(0, 0)\}$, show that the line integral of the one-form $\eta = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ depends on the choice of path from $(1, 0)$ to $(-1, 0)$.
- (c) Compute $d\omega$ and $d\eta$, and explain, in light of the theorem on independence of line integrals of closed forms along homotopic paths, why you may expect the line integrals chosen above in parts (a) and (b) to give different answers depending on the paths.
- (18) Let E be a vector bundle over a smooth manifold M , and let ∇_1 and ∇_2 be two connections on E . If f and g are smooth functions on M , find, with proof, necessary and sufficient conditions for $f\nabla_1 + g\nabla_2$ to be a connection on E .
- (19) Consider the hyperboloid model of hyperbolic space, with metric given by $g = \iota^* \lambda$ for ι the inclusion map and

$$\lambda = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2$$

is the standard Lorentzian metric on \mathbb{R}^{n+1} . Consider any smooth immersion $f: \Omega \rightarrow \mathbb{H}^n \subset \mathbb{R}^{n+1}$, where $\Omega \subset \mathbb{R}^n$ is a domain.

- (a) Using the method developed in class, compute the Christoffel symbols of the Levi-Civita connection of g in terms of λ .
- (b) Let $\gamma(t) = x \cosh t + v \sinh t$ be a geodesic in \mathbb{H}^n for $x, v \in \mathbb{R}^{n+1}$ with $\lambda(x, x) = -1$, $\lambda(x, v) = 0$, $\lambda(v, v) = 1$. If $w \in T_x \mathbb{H}^n$, compute the parallel transport of w along the geodesic γ .
- (c) Use the third covariant derivatives of f to show that the curvature of (\mathbb{H}^n, g) satisfies

$$R(X, Y)Z = -g(Z, Y)X + g(Z, X)Y.$$

- (20) Let M be a smooth manifold of dimension n , and let ∇ denote a connection on the tangent bundle TM . Consider the vector bundle E of rank $n + 1$ whose sections are given by pairs (v, h) ,

where v is a tangent vector field and h is a function. (We may write $E = TM \oplus \mathbb{R}$ in this case, as the *direct sum* of the rank- n vector bundle TM and the trivial line bundle \mathbb{R} .) Define an operator $\tilde{\nabla}$ on sections of E by

$$\tilde{\nabla}_X(v, h) = (\nabla_X v + hX, Xh)$$

for X any vector field on M .

- (a) Show $\tilde{\nabla}$ is a connection on E .
- (b) Show that $\tilde{\nabla}$ is flat (i.e., that its curvature vanishes identically) if and only if ∇ is flat and torsion-free.