(1) Let $\Gamma$ be a discrete group acting on a manifold $M$.
   (a) Define what it means for $\Gamma$ to act freely.
   (b) Define what it means for $\Gamma$ to act properly discontinuously.
   (c) Let $V$ be a finite-dimensional real vector space equipped with the usual topology. Let $v_1, v_2 \in V$. Let $\Gamma$ be the group $\mathbb{Z}^2$ with addition as the group law. If $\gamma = (n_1, n_2) \in \Gamma$, define the action of $\gamma$ on $V$ by
   \[ \gamma \cdot x = n_1 v_1 + n_2 v_2 + x. \]
   Find necessary and sufficient conditions on $\{v_1, v_2\}$ for the action of $\Gamma$ on $V$ to be free and properly discontinuous.

(2) (a) If $U$ is a manifold and $F = U \times \mathbb{R}^r$ is the trivial vector bundle, write down an explicit connection on $F$.
   (b) Let $M$ be a smooth manifold of dimension $n$, and let $E$ be a vector bundle of rank $r$ over $M$. Use a partition of unity argument to construct a connection over $E$.

(3) (a) Let $\mathcal{M}_n(\mathbb{C})$ denote the set of $n \times n$ complex-valued matrices. Show $\mathcal{M}_n(\mathbb{C})$ is a manifold of (real) dimension $2n^2$.
   (b) Let $f : \mathcal{M}_n(\mathbb{C}) \to V$ be defined by $f(A) = A \overline{A}^\top$, and $V = \{B \in \mathcal{M}_n(\mathbb{C}) : B = \overline{B}^\top\}$. Show $f$ is a submersion at each $A \in U(n)$, where $U(n) = \{A \in \mathcal{M}_n(\mathbb{C}) : A \overline{A}^\top = I\}$.
   (c) Show $U(n)$ is a Lie group. Find its dimension.
   (d) Calculate the tangent space $T_I U(n)$.

(4) Let $(M, g)$ be a compact Riemannian manifold. If $\omega = \omega_i \, dx^i$ is a one-form, define $\|\omega\|^2_g = g^{ij} \omega_i \omega_j$ for $g^{ij}$ the inverse matrix of $g_{ij}$. Let $f$ be a function on $M$. Define
   \[ E_g(f) = \int_M \|df\|^2_g \, dV_g \]
   for $dV_g$ the volume density of the metric $g$. Let $u$ be a positive smooth function. Assume the dimension of $M$ is 2. Show that $E_{u g}(f) = E_g(f)$.

(5) Recall a differential form $\eta$ on a manifold $M$ is called closed if $d\eta = 0$.
   (a) Let $M$ be a simply connected manifold, let $\eta$ be a closed one-form on $M$, and let $p \in M$. Show that
   \[ f(q) = \int_{[a,b]} \gamma^* \eta, \]
is well-defined independently of the interval \([a, b]\) and the smooth path \(\gamma: [a, b] \rightarrow M\) from \(p\) to \(q\). (Feel free to use any result proved in class.)

(b) Under the assumptions of part (a), show that \(\eta = df\). (Hint: In local coordinates, consider paths near \(q\) in each coordinate direction.)

(6) (a) Give, without proof, an example of a connected Riemannian manifold \((M, g)\) and points \(p, q\) in \(M\) so that there is no path from \(p\) to \(q\) of length equal to the distance from \(p\) to \(q\).

(b) Give, without proof, an example of a complete Riemannian manifold \((M, g)\) and points \(p, q \in M\) so that there is more than one geodesic from \(p\) to \(q\).

(c) Give, without proof, an example of a Riemannian manifold \((M, g)\) and a complete geodesic curve which is dense in \(M\).