Discrete Structures: Sample Questions, Exam 1

1. Prove by mathematical induction that $3 \mid (n^3 - n)$ for every positive integer n.

2. Write down a formula for the sequence

$$3, 4, 6, 9, 13, 18, 24, 31, 38, \ldots$$

   Is your formula recursive or explicit?

3. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

   Compute $\mathbf{A} \odot \mathbf{B}$, $\mathbf{AB}$, $\mathbf{B} \odot \mathbf{A}$ and $\mathbf{A} \wedge \mathbf{B}$.

   Assume $\mathbf{A}$ is the matrix of a relation. Draw the corresponding digraph.

4. The following arrays describe a relation R on the set $A = \{1, 2, 3, 4, 5\}$. Compute both the digraph of R and the matrix $\mathbf{M}_R$.

$$\begin{aligned} \text{VERT} &= [6, 2, 8, 7, 10] \\ \text{TAIL} &= [2, 2, 2, 2, 1, 1, 4, 3, 4, 5] \\ \text{HEAD} &= [4, 3, 5, 1, 2, 3, 5, 4, 2, 4] \\ \text{NEXT} &= [3, 1, 4, 0, 0, 5, 9, 0, 0, 0] \end{aligned}$$

5. Given the partition $\mathcal{P} = \{\{1, 2\}, \{3\}, \{4, 5\}\}$ of the set $A = \{1, 2, 3, 4, 5\}$, consider R the associated equivalence relation on A. Draw the digraph associated to R and write down the matrix $\mathbf{M}_R$.

6. Use the Euclidean algorithm to compute the greatest common divisor GCD(75, 12). Show your work. Compute the least common multiple LCM(75, 12).

7. How many ways can a committee of 3 faculty members and 2 students be selected from 7 faculty members and 8 students? Show your work.

8. True/False. Circle T or F. No explanation needed.

(a)	T	F	If $\mathbf{A}$ and $\mathbf{B}$ are any 2×2 matrices, then $\mathbf{AB} = \mathbf{BA}$.
(b)	T	F	If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are 2×2 matrices, then $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.
(c)	T	F	Let $A = Z^+$ the set of positive integers. Define the relation R on A by aRb if and only if $a \mid b$. R is transitive.
(d)	T	F	R the relation from (c) is symmetric.
(e)	T	F	R the relation from (c) is asymmetric.
(f)	T	F	R the relation from (c) is antisymmetric.
(g)	T	F	R the relation from (c) is irreflexive.
(h)	T	F	R the relation from (c) is an equivalence relation.
(i)	T	F	Let $R = (A, \$)$ be the mathematical structure where A is the set of even integers and \$ is a unary operation on A given by $\$a = \frac{a}{2}$ for every $a \in A$. A is closed with respect to the operation \$.
(j)	T	F	Given the digraph of a relation R on A, the digraph of the inverse relation R^{-1} is given by reversing the direction of each arrow in the digraph.
(k)	T	F	Given the digraph of a relation R on A, the digraph of complementary relation $\bar{R}$ is given by reversing the direction of each arrow in the digraph.
(l)	T	F	If R is a reflexive relation, then the connectivity relation R^∞ is equal to the reachability relation R^*.
(m)	T	F	49 and 77 are relatively prime.
(n)	T	F	Let A and B be subsets of a universal set U. Then it is always true that $\lvert A \cup B \rvert = \lvert A \rvert + \lvert B \rvert$.
(o)	T	F	Two cards are dealt in succession from a standard shuffled 52-card deck. The number of possible 2-card hands is 1326.

9. Let a_n be the sequence recursively defined by $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1}a_{n-2}$ for $n \geq 3$. Compute the first five terms $a_1, \ldots, a_5$.

10. Consider the relation on $A = \{a, b, c, d, e\}$.

$$R = \{(a, a), (a, c), (b, c), (c, d), (e, a), (d, b)\}.$$

Draw the corresponding digraph. Use Warshall's algorithm to compute the matrix $\mathbf{M}_{R^\infty}$. Show all the intervening steps.

11. A 6-sided die is rolled twice. What is the probability that the sum of the two rolls is exactly 8?