Discrete Structures: Sample Questions, Exam 1, Solutions

1. Prove by mathematical induction that $3 \mid (n^3 - n)$ for every positive integer n.

   Answer: Basis step $n = 1$: $n^3 - n = 1^3 - 1 = 0$, and so we must check that $3 \mid 0$. This is true since $0 = 3(0)$.

   Induction step. For $k \geq 1$, assume $P(k)$, which is that $3 \mid (k^3 - k)$. We want to show $P(k+1)$. In other words, we want to show that $3 \mid [(k+1)^3 - (k+1)]$.

   So by $P(k)$, we know that there is an integer s satisfying

   $$P(k): \qquad k^3 - k = 3s.$$

   To show $P(k+1)$, compute

   $$\begin{aligned} [(k+1)^3 - (k+1)] &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + k + (3k^2 + 2k) \\ &= 3s + 3(k^2 + k) \qquad \text{by } P(k) \\ &= 3(s + k^2 + k). \end{aligned}$$

   Therefore $3 \mid [(k+1)^3 - (k+1)]$, and we have shown $P(k+1)$.

   Note to get from the second line to the third in the computation, we want to use $P(k)$. So therefore, in the second line, we want to find the expression $k^3 - k$. This is the motivation in going from the second to the third lines.

   Since we have the basis step $n = 1$ and the induction step, the proof is complete.

2. Write down a formula for the sequence

   $$3, 4, 6, 9, 13, 18, 24, 31, 38, \ldots$$

   Is your formula recursive or explicit?

   Answer: $a_1 = 3$. $a_n = a_{n-1} + (n-1)$. This is a recursive formula.

3. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Compute $\mathbf{A} \odot \mathbf{B}$, $\mathbf{AB}$, $\mathbf{B} \odot \mathbf{A}$ and $\mathbf{A} \wedge \mathbf{B}$.

Assume $\mathbf{A}$ is the matrix of a relation. Draw the corresponding digraph.

Answer:

$$\begin{aligned} \mathbf{A} \odot \mathbf{B} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ \mathbf{AB} &= \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ \mathbf{B} \odot \mathbf{A} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ \mathbf{A} \wedge \mathbf{B} &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

For the digraph, see the picture file.

4. The following arrays describe a relation R on the set $A = \{1, 2, 3, 4, 5\}$. Compute both the digraph of R and the matrix $\mathbf{M}_R$.

$$\begin{aligned} \text{VERT} &= [6, 2, 8, 7, 10] \\ \text{TAIL} &= [2, 2, 2, 2, 1, 1, 4, 3, 4, 5] \\ \text{HEAD} &= [4, 3, 5, 1, 2, 3, 5, 4, 2, 4] \\ \text{NEXT} &= [3, 1, 4, 0, 0, 5, 9, 0, 0, 0] \end{aligned}$$

Answer:

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

For the digraph, see the picture file.

5. Given the partition $\mathcal{P} = \{\{1,2\},\{3\},\{4,5\}\}$ of the set $A = \{1,2,3,4,5\}$, consider R the associated equivalence relation on A. Draw the digraph associated to R and write down the matrix $\mathbf{M}_R$.

   Answer:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

   For the digraph see the picture file.

6. Use the Euclidean algorithm to compute the greatest common divisor GCD(75, 12). Show your work. Compute the least common multiple LCM(75, 12).

   Answer:

$$\begin{aligned} 75 &= 6(12) + 3 \\ 12 &= 4(3) + 0 \end{aligned}$$

   So the last one before the 0 is 3. So $\text{GCD}(75,12) = 3$.

$$\text{LCM}(75,12) = \frac{(75)(12)}{\text{GCD}(75,12)} = \frac{(75)(12)}{3} = (75)4 = 300.$$

7. How many ways can a committee of 3 faculty members and 2 students be selected from 7 faculty members and 8 students? Show your work.

   Answer: Task T_1: to choose 3 faculty members from 7, there are

$${}_7C_3 = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

   ways. Task T_2: to choose 2 students from 8, there are

$${}_8C_2 = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

   ways. All together, there are $35 \cdot 28 = 980$ ways of choosing this committee.

8. True/False. Circle T or F. No explanation needed.

(a) T F If **A** and **B** are any 2×2 matrices, then $\mathbf{AB} = \mathbf{BA}$.
Answer: F—Matrix multiplication is not in general commutative.

(b) T F If **A**, **B** and **C** are 2×2 matrices, then $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.
Answer: T—Matrix multiplication is associative.

(c) T F Let $A = Z^+$ the set of positive integers. Define the relation R on A by aRb if and only if $a \mid b$. R is transitive.
Answer: T—If $a \mid b$ and $b \mid c$, then $b = sa$ and $c = tb$ for $s, t \in Z$. Therefore, $c = tb = t(sa) = (ts)a$ and so $a \mid c$.

(d) T F R the relation from (c) is symmetric.
Answer: F—$2 \mid 4$ but $4 \not\mid 2$, so this is not always true.

(e) T F R the relation from (c) is asymmetric.
Answer: F—The statement

$$a \mid b \Longrightarrow b \not\mid a$$

is not true in the case $a = b$.

(f) T F R the relation from (c) is antisymmetric.
Answer: T—If $a \mid b$ and $b \mid a$, then $a = b$. This is kind of obvious, but a proof goes like this: Since $a \mid b$, $b = sa$ for $s \in Z^+$. Similarly, $a = tb$ for $t \in Z^+$ since $b \mid a$. Therefore, $a = tb = t(sa) = (ts)a$, and so $ts = 1$. Since t and s are both positive integers, they must both equal 1. Therefore, $a = tb = (1)b = b$.

(g) T F R the relation from (c) is irreflexive.
Answer: F—In fact $a \not\mid a$ is false for all $a \in Z^+$.

(h) T F R the relation from (c) is an equivalence relation.
Answer: F—It is not symmetric (though it is transitive and reflexive).

(i) T F Let $R = (A, \$)$ be the mathematical structure where A is the set of even integers and \$ is a unary operation on A given by $\$a = \frac{a}{2}$ for every $a \in A$. A is closed with respect to the operation \$.
Answer: F—For $a = 2$, $\$a = 1 \notin A$.

(j) T F Given the digraph of a relation R on A, the digraph of the inverse relation R^{-1} is given by reversing the direction of each arrow in the digraph.

Answer: T—$bR^{-1}a$ exactly when aRb. This amounts to reversing the arrows in the digraph.

(k) T F Given the digraph of a relation R on A, the digraph of complementary relation $\bar{R}$ is given by reversing the direction of each arrow in the digraph.

Answer: F—The relation $\bar{R}$ is formed by erasing all the arrows in the digraph and putting in all the arrows which were not originally there.

(l) T F If R is a reflexive relation, then the connectivity relation R^∞ is equal to the reachability relation R^*.

Answer: T—If R is reflexive, then it must contain every pair (a, a). Recall R^∞ is the relation given by $aR^\infty b$ if and only if there is a path of positive length from a to b. R^* is the relation given by aR^*b if and only if $aR^\infty b$ or $a = b$. Since R is reflexive, there is a path of length 1 from each a to itself. So $aR^\infty a$ for all $a \in A$, and so the extra condition in R^* is automatically satisfied already in R^∞ and $R^* = R^\infty$.

(m) T F 49 and 77 are relatively prime.

Answer: F—$\text{GCD}(49, 77) = 7 \neq 1$.

(n) T F Let A and B be subsets of a universal set U. Then it is always true that $|A \cup B| = |A| + |B|$.

Answer: F—In general $|A \cup B| = |A| + |B| - |A \cap B|$, and so the equation is false if A and B are not disjoint.

(o) T F Two cards are dealt in succession from a standard shuffled 52-card deck. The number of possible 2-card hands is 1326.

Answer: T—The number of such hands is

$$_{52}C_2 = \frac{52 \cdot 51}{2 \cdot 1} = 1326.$$

9. Let a_n be the sequence recursively defined by $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1}a_{n-2}$ for $n \geq 3$. Compute the first five terms $a_1, \ldots, a_5$.

Answer:

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 3 \\ a_3 &= a_2 a_1 = 2(3) = 6 && \text{(use } n = 3) \\ a_4 &= a_3 a_2 = 6(3) = 18 \\ a_5 &= a_4 a_3 = 18(6) = 108 \end{aligned}$$

10. Consider the relation on $A = \{a, b, c, d, e\}$.

$$R = \{(a, a), (a, c), (b, c), (c, d), (e, a), (d, b)\}.$$

Draw the corresponding digraph. Use Warshall's algorithm to compute the matrix $\mathbf{M}_{R^\infty}$. Show all the intervening steps.

Answer:

For the digraph, see the picture file. Compute

$$\mathbf{W}_0 = \mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{W}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$
\begin{aligned}
\mathbf{W}_3 &= \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \\
\mathbf{W}_4 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \\
\mathbf{M}_{R^\infty} = \mathbf{W}_5 &= \mathbf{W}_4.
\end{aligned}
$$

11. A 6-sided die is rolled twice. What is the probability that the sum of the two rolls is exactly 8?

    Answer: The sample space for two rolls of a die is

    $$A = \{(1,1),(1,2),\ldots,(6,5),(6,6)\},$$

    and $|A| = 6^2 = 36$. The event given by the sum of the two rolls being 8 is given by

    $$E = \{(2,6),(3,5),(4,4),(5,3),(6,2)\},$$

    and so $|E| = 5$. So the probability is

    $$\frac{|E|}{|A|} = \frac{5}{36}.$$