Discrete Structures: Solutions to Sample Questions, Exam 2

1. Let $A = B = \{a, b, c\}$. Consider the relation $g = \{(a, b), (b, c), (c, c)\}$. Is g one-to-one? Is g onto? Why?

   Solution: g is not one-to-one, since for $c \in A$, $g(b) = g(c) = c$. g is not onto, since $a \notin g(A)$.

2. Consider $f : Z^+ \to Z^+$ defined by $f(a) = a^2$. Is f one-to-one? Is f onto? Why?

   Solution: f is one-to-one, since if $f(a) = f(b)$ for $a, b \in Z^+$, then $a^2 = b^2$. Therefore, since elements of Z^+ are positive, then we have $a = b$ and thus f is one-to-one. f is not onto. For example, 2 is not in the image $f(Z^+)$.

3. Put the following functions in order from lowest to highest in terms of their Θ classes. (Some of the functions may be in the same Θ class. Indicate that on your list also.)

   (a) $f_1(n) = n \log n$,

   (b) $f_2(n) = n^{\frac{3}{2}}$,

   (c) $f_3(n) = 10{,}000$,

   (d) $f_4(n) = \sqrt{n}(n + \log n)$,

   (e) $f_5(n) = 3^n$,

   (f) $f_6(n) = 2^{n+2}$,

   (g) $f_7(n) = 0.0001$.

   Solution: These are listed from lowest to highest, with functions with the same Θ class listed on the same line:

   - $f_3(n) = 10{,}000$, $f_7(n) = 0.0001$.
   - $f_1(n) = n \log n$.
   - $f_2(n) = n^{\frac{3}{2}}$, $f_4(n) = \sqrt{n}(n + \log n)$.
   - $f_6(n) = 2^{n+2}$.

- $f_5(n) = 3^n$.

Why? $f_3(n)$ and $f_7(n)$ are both constants, which is the lowest order of growth.

$f_1(n) = n \log n$, while $f_2(n) = n^{\frac{3}{2}}$. $f_1(n)$ has lower growth order since $f_1(n) = n \cdot \log n$ and $f_2(n) = n \cdot n^{\frac{1}{2}}$ and $\log n$ grows more slowly than $n^{\frac{1}{2}}$ (and indeed $\log n$ grows more slowly than any positive power of n).

$f_2(n)$ and $f_4(n)$ are in the same Θ class, since $n + \log n$ is in the Θ class of n. (This is since $\log n$ has slower order of growth than n.) Therefore, $f_4(n) = \sqrt{n}(n + \log n)$ has the same Θ class as $\sqrt{n} \cdot n = n^{\frac{3}{2}}$.

$f_6(n)$ and $f_5(n)$ are both of higher Θ class than $n^{\frac{3}{2}}$, since any exponential function with base > 1 grows faster than any power of n. $f_5(n) = 3^n$ has higher Θ class than $f_6(n) = 2^{n+2}$, since $f_6(n) = 4 \cdot 2^n$ has the same Θ class as 2^n, and the Θ class of 3^n is higher than that of 2^n since the bases $3 > 2$.

4. Let $S = \{x, y, z\}$, and consider the set $P(S)$ with relation R given by set inclusion. Is R a partial order? Why or why not? (Carefully check the conditions needed for a relation to be a partial order.) Is R a linear order? Again carefully check the conditions for R to be a linear order.

   Solution: R is a partial order (i.e., R is reflexive, antisymmetric, and transitive). Recall elements $A, B \in P(S)$ are subsets $A, B \subseteq S$. Then, ARB if and only if $A \subseteq B$. R is reflexive since $A \subseteq A$ always. R is antisymmetric since if $A \subseteq B$ and $B \subseteq A$, then $A = B$. R is transitive since if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

   R is not a linear order. In a linear order, each two elements are comparable. In particular $\{x\}, \{y\} \in P(S)$ are not comparable, so that R is not a linear order.

5. Show that $P(S), R$ from the previous problem is isomorphic to the poset D_{42} of divisors of 42 with relation given by divisibility.

   Solution 1: Here is a quick proof: Since S has 3 elements $(P(S), \subseteq)$ is isomorphic to the Boolean algebra B_3. Since $42 = 2 \cdot 3 \cdot 7$ is the product of 3 distinct primes, then D_{42} is also isomorphic to B_3. It follows that $(P(S), \subseteq)$ and D_{42} are isomorphic since the condition of 2 posets being isomorphic is an equivalence relation (and is thus transitive).

Solution 2: One can also draw the Hasse diagrams (see the picture page). An explicit one-to-one correspondence that preserves the partial orders is the following (there are other such correspondences as well):

$$\begin{array}{cccc} \{x,y,z\} \leftrightarrow 42, & \{x,y\} \leftrightarrow 6, & \{x,z\} \leftrightarrow 14, & \{y,z\} \leftrightarrow 21, \\ \{x\} \leftrightarrow 2, & \{y\} \leftrightarrow 3, & \{z\} \leftrightarrow 7, & \emptyset \leftrightarrow 1. \end{array}$$

6. Is there an infinite poset with a least element? Either write down an example or prove that this is impossible.

   Solution: $(Z^+, \leq)$ is an infinite poset with least element 1.

7. Which of the following Hasse diagrams represent lattices?

   Solution: (I), (III) and (IV) are lattices, for each pair of elements has a least upper bound and a greatest lower bound. (This fails in (II), since a and b do not have a least upper bound.)

8. For the examples in the previous problem, which represent finite Boolean algebras?

   Solution: Only (I) represents a finite Boolean algebra. (It is isomorphic to B_2.) (II) does not since it is not even a lattice. (III) does not since it has $5 \neq 2^n$ elements. (IV) is not since there are no complements to each element: The unit element $I = f$ and the zero element $0 = i$ clearly. But given h, for example, there is no complement element h' which satisfies $\text{GLB}(h, h') = 0$ and $\text{LUB}(h, h') = I$: Check each element: $\text{GLB}(f, h) = \text{GLB}(g, h) = \text{GLB}(h, h) = h \neq 0$, while $\text{LUB}(h, i) = h \neq I$.

9. Draw the Hasse diagram for the lattice D_{18} consisting of the divisors of 18 with the partial order of divisibility.

   Solution: See picture page.

10. Given the following Karnaugh map of a Boolean function, write the function as an equivalent Boolean polynomial.

    Solution: $(y \wedge z) \vee (x \wedge y')$. See picture page for the corresponding rectangles.

11. Construct the labeled tree of the algebraic expression

$$(((x - y) * z) - 3)/(19 + (x * x)).$$

Solution: See picture page.

12. Let x, y, z be Boolean variables. Use the rules of Boolean algebra (or a truth table with Karnaugh map) to simplify the following expression. Your answer should be as simple as possible.

$$(x \wedge z) \vee (y' \vee (y' \wedge z)) \vee ((x \wedge y') \wedge z')$$

Solution 1: Let $f = (x \wedge z) \vee (y' \vee (y' \wedge z)) \vee ((x \wedge y') \wedge z')$ and compute

$$\begin{aligned}
f &= (x \wedge z) \vee y' \vee (x \wedge y' \wedge z') \qquad [\text{Simplify } y' \vee (y' \wedge z)] \\
&= y' \vee (x \wedge z) \vee (x \wedge y' \wedge z') \qquad [\vee \text{ is commutative}] \\
&= y' \vee (x \wedge (z \vee (y' \wedge z'))) \qquad [\wedge \text{ is distributive over } \vee] \\
&= y' \vee (x \wedge ((z \vee y') \wedge (z \vee z'))) \qquad [\vee \text{ is distributive over } \wedge] \\
&= y' \vee (x \wedge ((z \vee y') \wedge I)) \\
&= y' \vee (x \wedge (z \vee y')) \\
&= (y' \vee x) \wedge (y' \vee (z \vee y')) \qquad [\vee \text{ is distributive over } \wedge] \\
&= (y' \vee x) \wedge ((y' \vee y') \vee z) \qquad [\vee \text{ is commutative and associative}] \\
&= (y' \vee x) \wedge (y' \vee z) \\
&= y' \vee (x \wedge z) \qquad [\vee \text{ is distributive over } \wedge]
\end{aligned}$$

Solution 2: Again let $f = (x \wedge z) \vee (y' \vee (y' \wedge z)) \vee ((x \wedge y') \wedge z')$ and compute the truth table

x	y	z	y'	z'	$x \wedge z$	$y' \wedge z$	$y' \vee (y' \wedge z)$	$x \wedge y'$	$(x \wedge y') \wedge z'$	f
0	0	0	1	1	0	0	1	0	0	1
0	0	1	1	0	0	1	1	0	0	1
0	1	0	0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	1	1	1	1
1	0	1	1	0	1	1	1	1	0	1
1	1	0	0	1	0	0	0	0	0	0
1	1	1	0	0	1	0	0	0	0	1

From the associate Karnaugh map (see the picture page), we see that $f = y' \vee (x \wedge z)$.

13. If $b = 1011 \in B_4$, write down the minterm E_b in terms of the Boolean variables x_1, x_2, x_3, x_4.

    Solution: $x_1 \wedge x_2' \wedge x_3 \wedge x_4$.

14. True/False. Circle T or F. No explanation needed.

(a)	T	F	If f is a one-to-one function from an infinite set A to itself, then f must be onto. **Solution:** F. Problem 2 provides a counterexample.
(b)	T	F	If g is a one-to-one function from a finite set A to itself, then g must be onto. **Solution:** T. Use Theorem 4 in Section 5.1.
(c)	T	F	If (T, v_0) is a rooted tree on a set A, then the relation T is irreflexive. **Solution:** T. Theorem 2 in Section 7.1 (Otherwise, if vTv for some vertex v this creates a nonunique path from v_0 to v.)
(d)	T	F	If $A = \{1, 2, 3, 4, 5, 6\}$ and R is the relation $\{(1, 2), (1, 4), (3, 5), (3, 6)\}$, then R is a tree on A. **Solution:** F. If R were a tree, then the root would be the unique element with in-degree 0. In this case, both 1 and 3 have in-degree 0, so R cannot be a tree.
(e)	T	F	If x is a Boolean variable, then $x \vee x = x$. **Solution:** T.
(f)	T	F	If y is a Boolean variable, then $y \vee I = y$. **Solution:** F. $y \vee I = I$.
(g)	T	F	Let $B = \{0, 1\}$ with the standard partial order and let $A = B \times B \times B$ with the product partial order. Then A is isomorphic as a lattice to D_{60}. **Solution:** F. A is a Boolean algebra, while D_{60} is not (this is because $60 = 2^2 \cdot 3 \cdot 5$ has the prime 2 repeated in its prime decomposition).
(h)	T	F	D_{85} is a Boolean algebra. **Solution:** T. $85 = 5 \cdot 17$ is a product of distinct primes.
(i)	T	F	Every finite lattice has a least element. **Solution:** T. If the elements of the lattice are $a_1, \ldots, a_n$, then $a_1 \wedge \cdots \wedge a_n$ is the least element.

(j) T F Every poset has a greatest element.
Solution: F. For example, the poset $(\mathbb{R}, \leq)$ has no greatest element; i.e., there is no largest real number.

(k) T F $f(n) = \log_5(n)$ is $O(\lg(n))$.
Solution: T. $\log_5(n) = \lg(n)/\lg(5)$.

(l) T F All the vertices of a complete binary tree have out-degree either 0 or 2.
Solution: T. Every vertex of a complete 2-tree which is not a leaf (out-degree 0) has out-degree exactly 2.

(m) T F Every vertex of a tree has in-degree 1.
Solution: F. The root of a tree has in-degree 0, not 1.

(n) T F If g is the mod-10 function, then $g(405) = 4$.
Solution: F. $g(405) = 5$.

(o) T F If x and y are Boolean variables, then $(x \wedge y)' = x' \wedge y'$.
Solution: F. DeMorgan's Laws state that $(x \wedge y)' = x' \vee y'$.

15. (a) Draw the digraph of the relation R on the set A in the previous problem, part (d).

    Solution: See picture page.

    (b) Now take the digraph you've just drawn in the previous part and add a single arrow to form a new relation R' on A so that R' is a tree on A. Label the root of the tree R'.

    Solution: There are 2 possible solutions: Draw an arrow from 1 to 3; in this case, 1 is the root. Or draw an arrow from 3 to 1; in this case, 3 is the root.

16. Consider the rooted tree (T, v_0) given below.

    (a) List all the level-2 vertices of the tree.

    Solution: v_4, v_5, v_6, v_7.

    (b) List all the leaves of the tree.

    Solution: $v_1, v_{12}, v_9, v_5, v_{10}, v_{11}, v_7$.

    (c) List all the siblings of v_3.

    Solution: v_1, v_2.

    (d) Draw the digraph of the subtree $T(v_4)$ with root v_4.

    Solution: This is the subtree with vertices v_4, v_8, v_9, v_{12}.

17. Consider the relation R on $A = \{1, 2, 3, 4\}$ given by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Is R a partial order? Why or why not?

Solution: R is not a partial order since R is not antisymmetric ($2R3$ and $3R2$ but $2 \neq 3$.)

18. Consider the poset $(A, \leq)$ given by the following Hasse diagram.

(a) List all the minimal elements of $(A, \leq)$.
Solution: g.

(b) List all the upper bounds of $B = \{c, d\} \subseteq A$.
Solution: b, c.

(c) List a pair of incomparable elements of A (if such a pair exists).
Solution: a, c and d, e are incomparable pairs (you only need to write down one of the two pairs).

(d) Is $(A, \leq)$ a lattice? Why or why not?
Solution: $(A, \leq)$ is not a lattice since GLB(a, c) does not exist (d, e, f, g are all lower bounds, but there is no greatest element among them). Similarly, LUB(d, e) does not exist.

19. Construct the digraph of the positional binary tree for the following doubly linked list. Label each vertex with the corresponding data.

INDEX	LEFT	DATA	RIGHT
1	11		0
2	10	M	7
3	0	Q	0
4	8	T	0
5	3	V	4
6	0	X	2
7	0	K	0
8	0	D	0
9	6	G	5
10	0	C	0
11	9	Y	0

Solution: See picture page.

20. Let $A = \{*, q, 1\}$, with partial order determined by $q < 1 < *$. Put the following elements of $A \times A \times A$ in lexigraphic order:

$$(q, q, 1), \quad (q, 1, *), \quad (*, *, q), \quad (q, q, q), \quad (*, 1, 1), \quad (1, *, q), \quad (q, 1, 1), \quad (*, *, *)$$

Solution:

$$(q, q, q) \prec (q, q, 1) \prec (q, 1, 1) \prec (q, 1, *) \prec (1, *, q) \prec (*, 1, 1) \prec (*, *, q) \prec (*, *, *)$$