1. (12 pts) Short answer. Put your answer in the box. No partial credit.

(a) Compute \[
\begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
\end{bmatrix} \odot \begin{bmatrix}
  1 & 1 & 0 \\
  0 & 1 & 1 \\
\end{bmatrix}.
\]

\[
\begin{bmatrix}
  0 & 1 & 0 \\
  1 & 1 & 0 \\
\end{bmatrix}
\]

(b) Write the base-2 expression \((11010)_2\) as a decimal number.

\[
(11010)_2 = 1(2^4) + 1(2^3) + 0(2^2) + 1(2^1) + 0(2^0)
\]

\[
= 16 + 8 + 2 = 26
\]

(c) A sequence is defined recursively by \(a_1 = 5\), \(a_n = n^2 + a_{n-1}\) for \(n > 1\). What is \(a_3\)?

\[
a_1 = 5
\]

\[
n = 2: \quad a_2 = 2^2 + a_1 = 4 + 5 = 9
\]

\[
n = 3: \quad a_3 = 3^2 + a_2 = 9 + 9 = 18
\]

(d) How many words consisting of 2 distinct letters can be formed from the letters of the word LAMP? (Recall that we consider any string of letters a word, even if it is not a word in English.)

\[
\begin{aligned}
4 \cdot P_2 &= \frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 4 \cdot 3 = 12
\end{aligned}
\]

\[
\text{Or: The first letter can be chosen in one of 4 ways.}
\]

\[
\text{Once the first letter is chosen, there are 3 ways of choosing the second letter.}
\]

\[
\text{So the answer is } 4 \cdot 3 = 12
\]
2. (10 pts) Find the smallest positive integer \( n_0 \) which satisfies \( n! > 2^n \). Use mathematical induction to prove that \( n! > 2^n \) for all integers \( n \geq n_0 \).

Check:

\[
\begin{align*}
1! &= 1, & 2^1 &= 2 \\
2 &= 1 & 2^2 &= 4 \\
3! &= 2 \cdot 1 = 2, & 2^3 &= 8 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24, & 2^4 &= 16 \\
& & 24 \geq 16 & \text{and } n_0 = 4.
\end{align*}
\]

Induction proof:

for \( n_0 = 4 \), the Basis step is checked above.

Induction step:
Assume \( P(k) \): \( k! > 2^k \) for some \( k \geq n_0 = 4 \)

we want to show \( P(k+1) \): \( (k+1)! > 2^{k+1} \)

Note \( (k+1)! = (k+1)k! \) and \( 2^{k+1} = 2(2^k) \)

Therefore \( (k+1)! = (k+1)k! \)

\[
\begin{align*}
&> (k+1)2^k \quad \text{(use } P(k)) \\
&> 2(2^k) \quad \text{(use fact } k+1 > 2) \\
& \quad \text{(since } k \geq 4) \\
&= 2^{k+1}
\end{align*}
\]

So \( (k+1)! > 2^{k+1} \) and \( P(k+1) \) is proved.

This shows \( n! > 2^n \) for all \( n \geq n_0 = 4 \).
3. (10 pts) Consider the relation $R$ given by the matrix

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$ 

Use Warshall's algorithm to compute the matrix $M_{R^\infty}$. Show all the intervening steps.

$$W_0 = M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$k = 1$

Look at $1^{st}$ row of $W_0 = W_{k-1}$ 1's at $(k, j)$ for $j = 1, 3$, $k = 1$

1$^{st}$ column of $W_0$ 1's at $(i, k)$ $k = 1$

So for $W_1$, we need 1's at $(i, j) = (1, 1)$ and $(1, 3)$

So $W_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$k = 2$

2$^{nd}$ row of $W_1$: $(k, j)$ $k = 2$, $j = 3$

2$^{nd}$ column of $W_1$: $(i, k)$ $k = 2$, $i = 3$

So for $W_2$, we need a 1 at $(i, j) = (3, 3)$

$W_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$k = 3$

3$^{rd}$ row of $W_2$: $(k, j)$ $k = 3$, $j = 2, 3$

3$^{rd}$ column of $W_2$: $(i, k)$ $k = 3$, $i = 1, 2, 3$

So for $W_3$, we need 1's at all $(i, j)$ $i = 1, 2, 3$, $j = 2, 3$

So $W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = M_{R^\infty}$  

(stop at $k = 3$

Since is $3 \times 3$ matrix.)
4. (10 pts) An arrow on a wheel is spun. The arrow will come to rest on a number 1, 2, or 3, with each being equally likely. The arrow is spun twice, and the numbers recorded.

(a) (3 pts) Write down all the elements of the sample space.

\[ A = \{ (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3) \} \]

(b) (3 pts) What is the probability that two 1's are spun? Show your work.

The event \[ E = \{ (1,1) \} \]

So \[ P_e = \frac{|E|}{|A|} = \frac{1}{9} \]

(c) (4 pts) What is the probability that the sum of the two numbers spun is even? Show your work.

The sum of the 2 numbers being even is given by the event

\[ E = \{ (1,1), (1,3), (2,2), (3,1), (3,3) \} \]

So \[ P_e = \frac{|E|}{|A|} = \frac{5}{9} \]
5. (20 pts) True/False. Circle T or F. No explanation needed.

For parts (a)-(g) of the True-False, consider the relation $R$ given by the following digraph:

(a) $T$ $F$ $R$ is symmetric. For example, $1R2$ but $2 \not{R} 1$.
(b) $T$ $F$ $R$ is transitive. $3R4$ and $4R5$ but $3 \not{R} 5$.
(c) $T$ $F$ The connectivity relation $R^\infty$ is equal to the reachability relation $R^*$.
(d) $T$ $F$ $3R^5$. $3R^5$ since $3R4$ and $4R5$.
(e) $T$ $F$ $2R4$. $2R4$. Therefore $2 \not{R} 4$.
(f) $T$ $F$ The relative set $R(\{1, 2\})$ has more than 3 elements. $R(\{1, 2\}) = \{2, 3, 4, 5\}$
(g) $T$ $F$ The complementary relation of the reflexive closure of $R$ is equal to the reflexive closure of the complementary relation of $R$.
(h) $T$ $F$ If $A$ and $B$ are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$. De Morgan's Laws
(i) $T$ $F$ There is an event $E$ with probability $p(E) = 2$. All probabilities must be in $[0, 1]$.
(j) $T$ $F$ 91 and 169 are relatively prime. $\gcd(91, 169) = 13$

\[ \text{since } 91 = 7 \cdot 13, \quad 169 = 13^2. \]
\[ \text{(can be computed via Euclidean algorithm).} \]

(C): Recall $R^*$ is reflexive. But it is easy to see that $5 \not{R} 5$ (since no paths start at 5). So $R^* \neq R^\infty$.

(g) The reflexive closure of $R$ is reflexive of course. So then the complementary relation of the reflexive closure of $R$ is irreflexive. But of course the reflexive closure of $\overline{R}$ is reflexive. So the two cannot be equal.