1. (12 pts) Short answer. Put your answer in the box. No partial credit.

(a) If 11001 and 10101 represent elements of the Boolean algebra $B_5$, compute $(11001)' \lor (10101)$.

\[
(11001)' \lor (10101) = (00110) \lor (10101) = 10111
\]

(b) If $x$ and $y$ are Boolean variables and 1 is the unit element, simplify $((x' \lor y) \land y) \lor 1$. YOUR ANSWER SHOULD CONSIST OF A SINGLE SYMBOL.

\[
((x' \lor y) \land y) \lor 1 = y \lor 1 = 1
\]

More simply, any Boolean expression $x \lor 1 = 1$.

(c) If $f = \{(1, a), (2, c), (3, d), (4, d)\}$ is a function from $A = \{1, 2, 3, 4\}$ to $B = \{a, b, c, d\}$, compute $f(2)$.

\[
f(2) = c
\]

(d) List the leaves of the following tree:

\[
e, f, c,
\]

\[
i, j, k, l
\]
2. (10 pts) Let \((A, \leq)\) be a poset.

(a) (2 pts) Define what it means for \(a \in A\) to be a minimal element.
\[ a \in A \text{ is a minimal element if and only if} \]
\[ \text{there is no element } c \in A \text{ so that } c < a. \]

(b) (2 pts) Define what it means for \(a \in A\) to be a least element.
\[ a \in A \text{ is a least element if } a \leq x \text{ for all } x \in A. \]

(c) (6 pts) Let \(a \in A\) be a least element. Prove that this \(a\) is a minimal element. Moreover, prove that this \(a\) is the only minimal element in \(A\).

Let \(a \in A\) be a least element.
Therefore, for every \(x \in A\), \(a \leq x\). (\(\star\))
In particular, there is no element \(c \in A\) so that \(c < a\);
since \(c < a\) implies \(c \leq a\)
but we know by (\(\star\)) that \(a \leq c\).
So \(a = c\) by antisymmetric property.
This contradicts \(c < a\) though.

So there is no element \(c < a\).

This shows that \(a\) is a minimal element.

To show that this \(a\) is the only minimal element in \(A\),
assume \(a \in A\) is a least element, and let \(b \in A\)
be a minimal element. We want to show \(a = b\).

Since \(a\) is a least element, \(a \leq x\) for all \(x \in A\). (\(\star\))
Since \(b\) is a minimal element, there is no element \(c \in A\) so that \(c < b\). (\(\star\)\(\star\))
Take \(x = b\) in (\(\star\)) and so \(a \leq b\). Then (\(\star\)\(\star\)) implies that \(a \neq b\).
The only possibility is \(a = b\),
and so this \(a\) is the only minimal element.
3. (10 pts) Let $A = \{1, 2, 3\}, B = \{w, x, y, z\}$.

(a) If it is possible, write down a function $f: A \to B$ which is one-to-one. If it is not possible, explain why not.

For example, define $f$ by

\[f(1) = w, \ f(2) = y, \ f(3) = z\]

This $f$ is one-to-one, since no element of $B$ is hit more than once.

(b) If it is possible, write down a function $g: A \to B$ which is not one-to-one. If it is not possible, explain why not.

For example, define $g$ by

\[g(1) = w, \ g(2) = w, \ g(3) = z\]

This $g$ is not one-to-one since there are 2 elements 1, 2 which are both mapped to $w \in B$.

(c) If it is possible, write down a function $h: A \to B$ which is onto. If it is not possible, explain why not.

This is not possible, since $|A| = 3$ implies $|h(A)| \leq 3$ (each element of $A$ is sent to one element of $B$). But $|B| = 4$. Therefore, the image $h(A) \neq B$ and $h$ cannot be onto. (h is onto if and only if $h(A) = B$.)
4. (10 pts) Consider the following doubly linked list.

<table>
<thead>
<tr>
<th>index</th>
<th>LEFT</th>
<th>DATA</th>
<th>RIGHT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>y</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>+</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>÷</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>+</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>*</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>y</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>x</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) (5 pts) Write down the labeled positional binary tree associated to the above linked list. Recall that LEFT[1] points to the root of the tree.

(b) (5 pts) Write down the fully parenthesized mathematical expression encoded by the tree in part (a).

\[
(y + 2) \div ((x \neq y) + 3)
\]
5. (20 pts) True/False. Circle T or F. No explanation needed.

For parts (a)-(f) of the True-False, consider the partial order \( \leq \) on \( A = \{a, b, c, d, e, f\} \) given by the following Hasse diagram:

```
    f
   / \  \
  e   d  b
     /   \
    c   \
      /    a
```

(a) \( T \) \( F \) \((A, \leq)\) is a Boolean algebra. \( |A| = 6 \neq 2^n \).

(b) \( T \) \( F \) \( f \) is the greatest element of \((A, \leq)\). \( f \geq x \) for all \( x \in A \).

(c) \( T \) \( F \) \((A, \leq)\) is isomorphic to the lattice \( D_{20} \) consisting of the divisors of 20 with partial order given by divisibility. See below.

(d) \( T \) \( F \) \( \leq \) is a linear order on \( A \). \( e \) and \( d \) are incomparable.

(e) \( T \) \( F \) \( e > d \). \( e \) and \( d \) are incomparable.

(f) \( T \) \( F \) There are exactly 3 upper bounds of the set \( B = \{b, c\} \subseteq A \). The upper bounds of \( B = \{b, c\} \) are \( f, d \).

(g) \( T \) \( F \) If \( C = \{1, 2, 3\} \), then the relation \( R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\} \) is a partial order on \( C \). \( R \) is not transitive: \( 1R2 \) and \( 2R3 \) but \( 1R3 \).

(h) \( T \) \( F \) \( \Theta(3^n) \) is lower than \( \Theta(2^{2n}) \). \( 2^{2n} = (2^2)^n = 4^n \) and \( 3 < 4 \).

(i) \( T \) \( F \) A finite lattice must have a greatest element. See below.

(j) \( T \) \( F \) If \( B = \{0, 1\} \) with the standard partial order, then 011 \( \prec \) 101 as elements of \( B \times B \times B \) with lexicographic order. The first digits 0 \( < \) 1.

(c) \( D_{20} \) is clearly isomorphic to \((A, \leq)\).

(i) \( T \) \( F \) If \( a_1, \ldots, a_n \) are the elements of the lattice, then \( a_1, a_2, \ldots, a_n \) is the greatest element.