Discrete Structures: Solutions to Sample Questions, Exam 2

1. Let $A = B = \{a, b, c\}$. Consider the relation $g = \{(a, b), (b, c), (c, c)\}$. Is g one-to-one? Is g onto? Why?

   Solution: g is not one-to-one, since for $c \in A$, $g(b) = g(c) = c$. g is not onto, since $a \notin g(A)$.

2. Consider $f : Z^+ \to Z^+$ defined by $f(a) = a^2$. Is f one-to-one? Is f onto? Why?

   Solution: f is one-to-one, since if $f(a) = f(b)$ for $a, b \in Z^+$, then $a^2 = b^2$. Therefore, since elements of Z^+ are positive, then we have $a = b$ and thus f is one-to-one. f is not onto. For example, 2 is not in the image $f(Z^+)$.

3. Put the following functions in order from lowest to highest in terms of their Θ classes. (Some of the functions may be in the same Θ class. Indicate that on your list also.)

   (a) $f_1(n) = n \log n$,

   (b) $f_2(n) = n^{\frac{3}{2}}$,

   (c) $f_3(n) = 10{,}000$,

   (d) $f_4(n) = \sqrt{n}(n + \log n)$,

   (e) $f_5(n) = 3^n$,

   (f) $f_6(n) = 2^{n+2}$,

   (g) $f_7(n) = 0.0001$.

   Solution: These are listed from lowest to highest, with functions with the same Θ class listed on the same line:

   - $f_3(n) = 10{,}000$, $f_7(n) = 0.0001$.
   - $f_1(n) = n \log n$.
   - $f_2(n) = n^{\frac{3}{2}}$, $f_4(n) = \sqrt{n}(n + \log n)$.
   - $f_6(n) = 2^{n+2}$.

- $f_5(n) = 3^n$.

Why? $f_3(n)$ and $f_7(n)$ are both constants, which is the lowest order of growth.

$f_1(n) = n \log n$, while $f_2(n) = n^{\frac{3}{2}}$. $f_1(n)$ has lower growth order since $f_1(n) = n \cdot \log n$ and $f_2(n) = n \cdot n^{\frac{1}{2}}$ and $\log n$ grows more slowly than $n^{\frac{1}{2}}$ (and indeed $\log n$ grows more slowly than any positive power of n).

$f_2(n)$ and $f_4(n)$ are in the same Θ class, since $n + \log n$ is in the Θ class of n. (This is since $\log n$ has slower order of growth than n.) Therefore, $f_4(n) = \sqrt{n}(n + \log n)$ has the same Θ class as $\sqrt{n} \cdot n = n^{\frac{3}{2}}$.

$f_6(n)$ and $f_5(n)$ are both of higher Θ class than $n^{\frac{3}{2}}$, since any exponential function with base > 1 grows faster than any power of n. $f_5(n) = 3^n$ has higher Θ class than $f_6(n) = 2^{n+2}$, since $f_6(n) = 4 \cdot 2^n$ has the same Θ class as 2^n, and the Θ class of 3^n is higher than that of 2^n since the bases $3 > 2$.

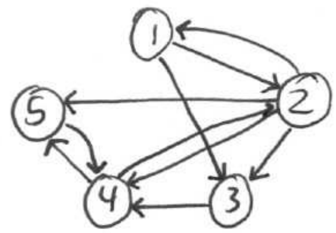
4. The following arrays describe a relation R on the set $A = \{1, 2, 3, 4, 5\}$. Compute both the digraph of R and the matrix $\mathbf{M}_R$.

$$\begin{aligned} \text{VERT} &= [6, 2, 8, 7, 10] \\ \text{TAIL} &= [2, 2, 2, 2, 1, 1, 4, 3, 4, 5] \\ \text{HEAD} &= [4, 3, 5, 1, 2, 3, 5, 4, 2, 4] \\ \text{NEXT} &= [3, 1, 4, 0, 0, 5, 9, 0, 0, 0] \end{aligned}$$

Answer:

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Digraph:

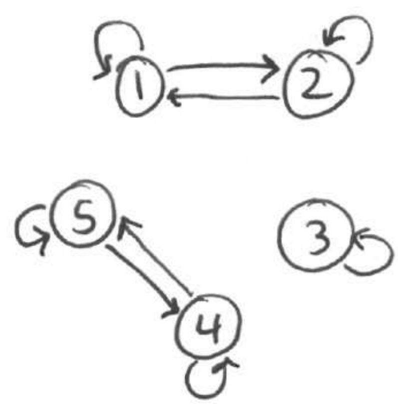


5. Given the partition $\mathcal{P} = \{\{1,2\},\{3\},\{4,5\}\}$ of the set $A = \{1,2,3,4,5\}$, consider R the associated equivalence relation on A. Draw the digraph associated to R and write down the matrix $\mathbf{M}_R$.

Answer:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Digraph:



6. Let $S = \{x, y, z\}$, and consider the set $P(S)$ with relation R given by set inclusion. Is R a partial order? Why or why not? (Carefully check

the conditions needed for a relation to be a partial order.) Is R a linear order? Again carefully check the conditions for R to be a linear order.

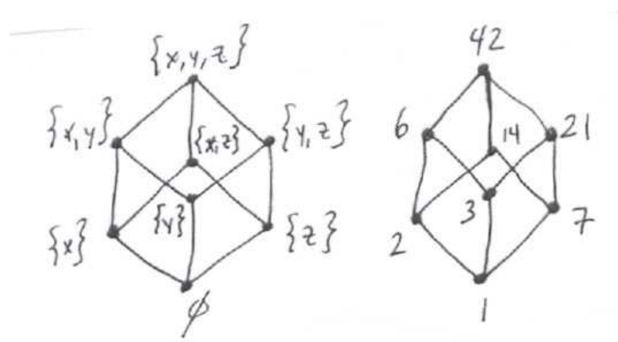
Solution: R is a partial order (i.e., R is reflexive, antisymmetric, and transitive). Recall elements $A, B \in P(S)$ are subsets $A, B \subseteq S$. Then, ARB if and only if $A \subseteq B$. R is reflexive since $A \subseteq A$ always. R is antisymmetric since if $A \subseteq B$ and $B \subseteq A$, then $A = B$. R is transitive since if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

R is not a linear order. In a linear order, each two elements are comparable. In particular $\{x\}, \{y\} \in P(S)$ are not comparable, so that R is not a linear order.

7. Show that $(P(S), R)$ from the previous problem is isomorphic to the poset D_{42} of divisors of 42 with relation given by divisibility.

   Solution 1: Here is a quick proof: Since S has 3 elements $(P(S), \subseteq)$ is isomorphic to the Boolean algebra B_3. Since $42 = 2 \cdot 3 \cdot 7$ is the product of 3 distinct primes, then D_{42} is also isomorphic to B_3. It follows that $(P(S), \subseteq)$ and D_{42} are isomorphic since the condition of 2 posets being isomorphic is an equivalence relation (and is thus transitive).

   Solution 2: One can also draw the Hasse diagrams:



   An explicit one-to-one correspondence that preserves the partial orders is the following (there are other such correspondences as well):

$$\begin{array}{cccc} \{x,y,z\} \leftrightarrow 42, & \{x,y\} \leftrightarrow 6, & \{x,z\} \leftrightarrow 14, & \{y,z\} \leftrightarrow 21, \\ \{x\} \leftrightarrow 2, & \{y\} \leftrightarrow 3, & \{z\} \leftrightarrow 7, & \emptyset \leftrightarrow 1. \end{array}$$

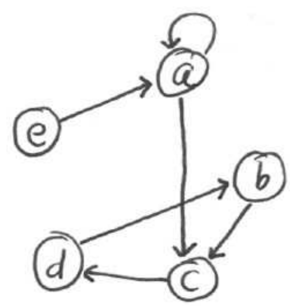
8. Consider the relation on $A = \{a, b, c, d, e\}$.

$$R = \{(a, a), (a, c), (b, c), (c, d), (e, a), (d, b)\}.$$

Draw the corresponding digraph. Use Warshall's algorithm to compute the matrix $\mathbf{M}_{R^\infty}$. Show all the intervening steps.

Answer:

Digraph:



Compute

$$\mathbf{W}_0 = \mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{W}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix},$$

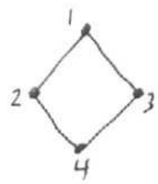
$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R^\infty} = \mathbf{W}_5 = \mathbf{W}_4.$$

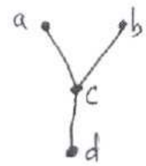
9. Is there an infinite poset with a least element? Either write down an example or prove that this is impossible.

   Solution: $(Z^+, \leq)$ is an infinite poset with least element 1.

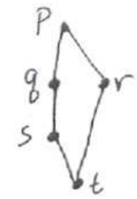
10. Which of the following Hasse diagrams represent lattices?



(I)



(II)



(III)

(IV)

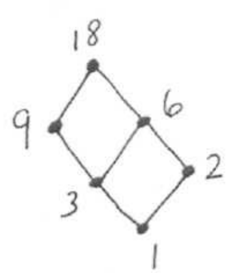
Solution: (I), (III) and (IV) are lattices, for each pair of elements has a least upper bound and a greatest lower bound. (This fails in (II), since a and b do not have a least upper bound.)

11. For the examples in the previous problem, which represent finite Boolean algebras?

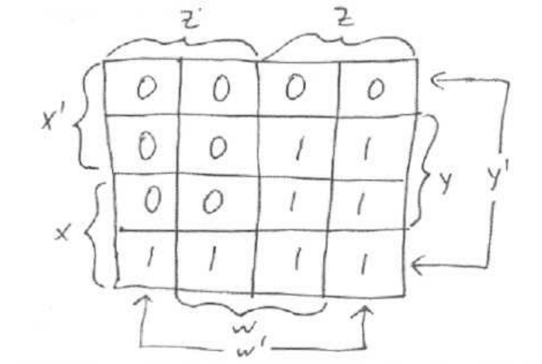
    Solution: Only (I) represents a finite Boolean algebra. (It is isomorphic to B_2.) (II) does not since it is not even a lattice. (III) does not since it has $5 \neq 2^n$ elements. (IV) is not since there are no complements to each element: The unit element $I = f$ and the zero element $0 = i$ clearly. But given h, for example, there is no complement element h' which satisfies $\mathrm{GLB}(h, h') = 0$ and $\mathrm{LUB}(h, h') = I$: Check each element: $\mathrm{GLB}(f, h) = \mathrm{GLB}(g, h) = \mathrm{GLB}(h, h) = h \neq 0$, while $\mathrm{LUB}(h, i) = h \neq I$.

12. Draw the Hasse diagram for the lattice D_{18} consisting of the divisors of 18 with the partial order of divisibility.

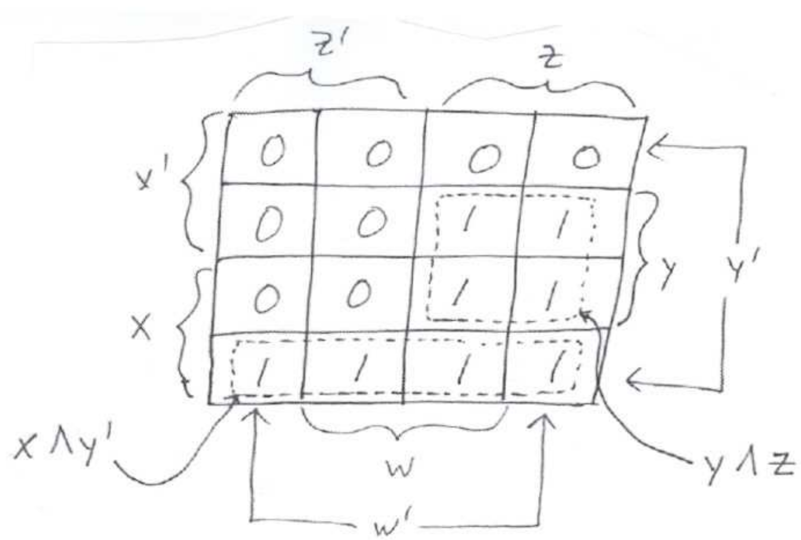
    Solution:



13. Given the following Karnaugh map of a Boolean function, write the function as an equivalent Boolean polynomial.



Solution: $(y \wedge z) \vee (x \wedge y')$. See below for the corresponding rectangles:



14. Let x, y, z be Boolean variables. Use the rules of Boolean algebra (or a truth table with Karnaugh map) to simplify the following expression. Your answer should be as simple as possible.

$$(x \wedge z) \vee (y' \vee (y' \wedge z)) \vee ((x \wedge y') \wedge z')$$

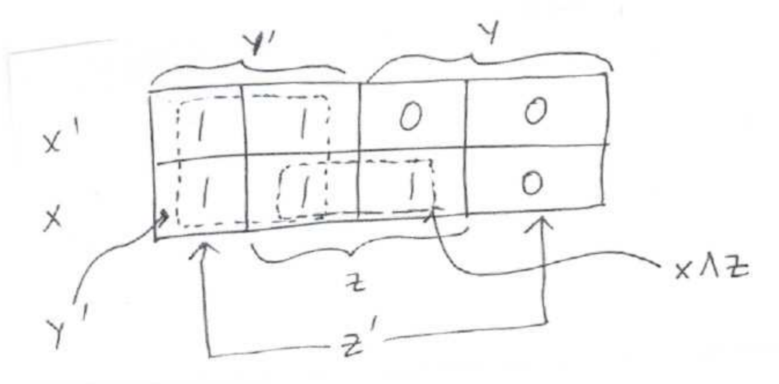
Solution 1: Let $f = (x \wedge z) \vee (y' \vee (y' \wedge z)) \vee ((x \wedge y') \wedge z')$ and compute

$$\begin{aligned}
f &= (x \wedge z) \vee y' \vee (x \wedge y' \wedge z') \qquad [\text{Simplify } y' \vee (y' \wedge z)] \\
&= y' \vee (x \wedge z) \vee (x \wedge y' \wedge z') \qquad [\vee \text{ is commutative}] \\
&= y' \vee (x \wedge (z \vee (y' \wedge z'))) \qquad [\wedge \text{ is distributive over } \vee] \\
&= y' \vee (x \wedge ((z \vee y') \wedge (z \vee z'))) \qquad [\vee \text{ is distributive over } \wedge] \\
&= y' \vee (x \wedge ((z \vee y') \wedge I)) \\
&= y' \vee (x \wedge (z \vee y')) \\
&= (y' \vee x) \wedge (y' \vee (z \vee y')) \qquad [\vee \text{ is distributive over } \wedge] \\
&= (y' \vee x) \wedge ((y' \vee y') \vee z) \qquad [\vee \text{ is commutative and associative}] \\
&= (y' \vee x) \wedge (y' \vee z) \\
&= y' \vee (x \wedge z) \qquad [\vee \text{ is distributive over } \wedge]
\end{aligned}$$

Solution 2: Again let $f = (x \wedge z) \vee (y' \vee (y' \wedge z)) \vee ((x \wedge y') \wedge z')$ and compute the truth table

x	y	z	y'	z'	$x \wedge z$	$y' \wedge z$	$y' \vee (y' \wedge z)$	$x \wedge y'$	$(x \wedge y') \wedge z'$	f
0	0	0	1	1	0	0	1	0	0	1
0	0	1	1	0	0	1	1	0	0	1
0	1	0	0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	1	1	1	1
1	0	1	1	0	1	1	1	1	0	1
1	1	0	0	1	0	0	0	0	0	0
1	1	1	0	0	1	0	0	0	0	1

From the associate Karnaugh map (see below), we see that $f = y' \vee (x \wedge z)$.



15. If $b = 1011 \in B_4$, write down the minterm E_b in terms of the Boolean variables x_1, x_2, x_3, x_4.

    Solution: $x_1 \wedge x_2' \wedge x_3 \wedge x_4$.

16. True/False. Circle T or F. No explanation needed.

(a) T F If f is a one-to-one function from an infinite set A to itself, then f must be onto.
Solution: F. Problem 2 provides a counterexample.

(b) T F If g is a one-to-one function from a finite set A to itself, then g must be onto.
Solution: T. Use Theorem 4 in Section 5.1.

(c) T F Let $A = Z^+$ the set of positive integers. Define the relation R on A by aRb if and only if $a|\, b$. R is transitive.
Answer: T—If $a|\, b$ and $b|\, c$, then $b = sa$ and $c = tb$ for $s, t \in Z$. Therefore, $c = tb = t(sa) = (ts)a$ and so $a|\, c$.

(d) T F R the relation from (c) is symmetric.
Answer: F—$2|\, 4$ but $4 \not| 2$, so this is not always true.

(e) T F R the relation from (c) is asymmetric.
Answer: F—The statement

$$a|\, b \Longrightarrow b \not| a$$

is not true in the case $a = b$.

(f) T F R the relation from (c) is antisymmetric.
Answer: T—If $a|\, b$ and $b|\, a$, then $a = b$. This is kind of obvious, but a proof goes like this: Since $a|\, b$, $b = sa$ for $s \in Z^+$. Similarly, $a = tb$ for $t \in Z^+$ since $b|\, a$. Therefore, $a = tb = t(sa) = (ts)a$, and so $ts = 1$. Since t and s are both positive integers, they must both equal 1. Therefore, $a = tb = (1)b = b$.

(g) T F R the relation from (c) is irreflexive.
Answer: F—In fact $a \not| a$ is false for all $a \in Z^+$.

(h) T F R the relation from (c) is an equivalence relation.
Answer: F—It is not symmetric (though it is transitive and reflexive).

(i) T F If x is a Boolean variable, then $x \vee x = x$.
Solution: T.

(j) T F If y is a Boolean variable, then $y \vee I = y$.
Solution: F. $y \vee I = I$.

(k) T F Let $B = \{0, 1\}$ with the standard partial order and let $A = B \times B \times B$ with the product partial order. Then A is isomorphic as a lattice to D_{60}.
Solution: F. A is a Boolean algebra, while D_{60} is not (this is because $60 = 2^2 \cdot 3 \cdot 5$ has the prime 2 repeated in its prime decomposition).

(l) T F D_{85} is a Boolean algebra.
Solution: T. $85 = 5 \cdot 17$ is a product of distinct primes.

(m) T F Every finite lattice has a least element.
Solution: T. If the elements of the lattice are $a_1, \ldots, a_n$, then $a_1 \wedge \cdots \wedge a_n$ is the least element.

(n) T F Every poset has a greatest element.
Solution: F. For example, the poset $(\mathbb{R}, \leq)$ has no greatest element; i.e., there is no largest real number.

(o) T F $f(n) = \log_5(n)$ is $O(\lg(n))$.
Solution: T. $\log_5(n) = \lg(n)/\lg(5)$.

(p) T F If g is the mod-10 function, then $g(405) = 4$.
Solution: F. $g(405) = 5$.

(q) T F If x and y are Boolean variables, then $(x \wedge y)' = x' \wedge y'$.
Solution: F. DeMorgan's Laws state that $(x \wedge y)' = x' \vee y'$.

(r) T F Given the digraph of a relation R on A, the digraph of the inverse relation R^{-1} is given by reversing the direction of each arrow in the digraph.
Answer: T—$bR^{-1}a$ exactly when aRb. This amounts to reversing the arrows in the digraph.

(s) T F Given the digraph of a relation R on A, the digraph of complementary relation $\bar{R}$ is given by reversing the direction of each arrow in the digraph.
Answer: F—The relation $\bar{R}$ is formed by erasing all the arrows in the digraph and putting in all the arrows which were not originally there.

(t) T F If R is a reflexive relation, then the connectivity relation R^∞ is equal to the reachability relation R^*.
Answer: T—If R is reflexive, then it must contain every pair (a, a). Recall R^∞ is the relation given by $aR^\infty b$ if and only if there is a path of positive length from a to b. R^* is the relation given by aR^*b if and only if $aR^\infty b$ or $a = b$. Since R is reflexive, there is a path of length 1 from each a to itself. So $aR^\infty a$ for all $a \in A$, and so the extra condition in R^* is automatically satisfied already in R^∞ and $R^* = R^\infty$.

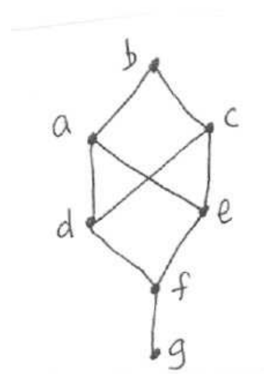
17. Consider the relation R on $A = \{1, 2, 3, 4\}$ given by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Is R a partial order? Why or why not?

Solution: R is not a partial order since R is not antisymmetric ($2R3$ and $3R2$ but $2 \neq 3$.)

18. Consider the poset $(A, \leq)$ given by the following Hasse diagram.



(a) List all the minimal elements of $(A, \leq)$.
Solution: g.

(b) List all the upper bounds of $B = \{c, d\} \subseteq A$.
Solution: b, c.

(c) List a pair of incomparable elements of A (if such a pair exists).
Solution: a, c and d, e are incomparable pairs (you only need to write down one of the two pairs).

(d) Is $(A, \leq)$ a lattice? Why or why not?
Solution: $(A, \leq)$ is not a lattice since $\text{GLB}(a, c)$ does not exist (d, e, f, g are all lower bounds, but there is no greatest element among them). Similarly, $\text{LUB}(d, e)$ does not exist.

19. Let $A = \{*, q, 1\}$, with partial order determined by $q < 1 < *$. Put the following elements of $A \times A \times A$ in lexigraphic order:

$(q, q, 1), \quad (q, 1, *), \quad (*, *, q), \quad (q, q, q), \quad (*, 1, 1), \quad (1, *, q), \quad (q, 1, 1), \quad (*, *, *)$

Solution:

$(q, q, q) \prec (q, q, 1) \prec (q, 1, 1) \prec (q, 1, *) \prec (1, *, q) \prec (*, 1, 1) \prec (*, *, q) \prec (*, *, *)$