1. (12 pts) Short answer. Put your answer in the box. No partial credit.

(a) Compute \(a_3\) for the sequence \(a_n\) defined recursively by

\[
a_1 = 1, \quad a_n = a_{n-1} + 5n \quad \text{for} \quad n \geq 2.
\]

**Solution:** \(a_3 = 26\). (Since \(a_2 = a_1 + 5(2) = 1 + 10 = 11, \ a_3 = a_2 + 5(3) = 11 + 15 = 26\).)

(b) Compute \(A \oplus B\) for

\[
A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.
\]

**Solution:**

\[
\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\]

(c) Compute the greatest common divisor \(\text{GCD}(50, 12)\).

**Solution:** 2. (Since \(50 = 2 \cdot 5^2, 12 = 2^2 \cdot 3\), so \(\text{GCD}(50, 12) = 2\). Or you can use the Euclidean algorithm: \(50 = 4(12) + 2, \ 12 = 6(2) + 0\), so the answer is 2.)

(d) In the box below, draw the digraph for the relation \(R\) on the set \(C = \{1, 2, 3\}\) given by

\(1R2, \ 1R3, \ 2R2\).

**Solution:**

![Diagram](image)

(e) What is the probability that a fair coin flipped once will come up “heads”?

**Solution:** 1/2.

(f) If \(D\) is a set containing exactly 4 distinct elements, how many distinct elements does the Cartesian product \(D \times D\) have?

**Solution:** \(|D \times D| = |D|^2 = 4^2 = 16\).
Solution: 12. There are 4 letters in data, 1 D, 2 A’s, and 1 T. The number of distinguishable permutations is then
\[
\frac{4!}{2! \cdot 1! \cdot 1!} = \frac{24}{2} = 12.
\]

3. (6 pts) Prove the following statement by mathematical induction: For each \( n \geq 1 \),
\[
P(n) : \quad 1 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.
\]

Solution: : Base case: \( n = 1 \).
\[
P(1) : \quad 1 + 2^1 \neq 2^{1+1} - 1
\]
is true since \( 1 + 2^1 = 3 = 2^{1+1} - 1 \).
Induction case: Assume \( k \geq 1 \) and assume
\[
P(k) : \quad 1 + 2^1 + \cdots + 2^k = 2^{k+1} - 1
\]
is true. Then we want to show
\[
P(k + 1) : \quad 1 + 2^1 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1
\]
is true. Compute using the left-hand side of \( P(k + 1) \)
\[
1 + 2^1 + \cdots + 2^k + 2^{k+1} = \left(1 + 2^1 + \cdots + 2^k\right) + 2^{k+1}
\]
\[
= (2^{k+1} - 1) + 2^{k+1} \quad \text{by } P(k)
\]
\[
= 2 \cdot 2^{k+1} - 1
\]
\[
= 2^{k+2} - 1
\]
\[
= 2^{(k+1)+1} - 1
\]
Thus \( P(k) \Rightarrow P(k + 1) \), and the induction case is proved. Thus \( P(n) \) is true for all \( n \geq 1 \).

4. (12 pts) Consider the Boolean matrix
\[
M_R = \begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]
as the matrix of a relation \( R \) from \( A = \{1, 2, 3, 4\} \) to \( B = \{x, y, z\} \).

(a) Write down the relation \( R \) as a subset of \( A \times B \). (In other words, write \( R \) as a set of ordered pairs in \( A \times B \).)

Solution: \( R = \{(1, x), (1, z), (3, y), (3, z), (4, x), (4, z)\} \).
(b) For the relation \( R \) above, what is the domain \( \text{Dom}(R) \)?

**Solution**: \( \text{Dom}(R) = \{1, 3, 4\} \).

(c) For the relation \( R \) above, what is the range \( \text{Ran}(R) \)?

**Solution**: \( \text{Ran}(R) = \{x, y, z\} = B \).

5. (4 pts) A password is to be written with 3 characters: The first is a lowercase letter, the second is a digit (chosen from 0 to 9), and the third may be either a lowercase letter or a digit. How many possible passwords can be formed in this way? Show your work. You do not have to simplify your answer.

**Solution**: \( 26 \cdot 10 \cdot 36 = 9360 \). We have three tasks to perform in succession: \( T_1, T_2, T_3 \) to pick the first, second, and third characters respectively. There are 26 ways to do \( T_1 \) (one for each letter), 10 ways to do \( T_2 \) (one for each of the 10 digits), and 36 ways to do \( T_3 \) (one for each letter + one for each digit). The total number of ways is the product \( 26 \cdot 10 \cdot 36 \).

6. (20 pts) True/False. Circle T or F. No explanation needed.

(a) T F 1 = 0! (where 0! means “zero factorial”)

**Solution**: T: 0! = 1 by definition (this is the base case of the recursive definition of \( n! \).)

(b) T F Consider logical statements \( p \) and \( q \). If \( q \) is false, and \( p \Rightarrow q \) is false, then \( p \) is ...

**Solution**: T: A glance at the truth table

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \Rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

shows that if \( p \Rightarrow q \) is \( F \), \( p \) must be \( T \).

(c) T F Matrix addition of \( 2 \times 2 \) matrices is commutative.

**Solution**: T:

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
+ \begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix}
= \begin{bmatrix}
a_{11} + b_{11} & a_{12} + b_{12} \\
a_{21} + b_{21} & a_{22} + b_{22}
\end{bmatrix}
= \begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix}
+ \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\]
(d) T F Matrix multiplication of $2 \times 2$ matrices is commutative.
Solution: F: For example
\[
\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} 
ot= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]

(e) T F The combination $5C_3$ is an even integer.
Solution: T: $5C_3 = \frac{5!}{3!2!} = 10$.

(f) T F If $C = \{a, b, c, a, x\}$, then the cardinality $|C| = 4$.
Solution: T: $C = \{a, b, c, x\}$ has 4 elements.

(g) T F If $A$ and $B$ are sets, then $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
Solution: F: DeMorgan’s Law states that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

(h) T F Consider the relation $R$ from $\{1, 2, 3\}$ to $\{a, b, c\}$ given by
\[
R = \{(1, b), (1, c), (2, a), (2, b)\}.
\]
Then the relative set $R(3) = \emptyset$.
Solution: T: $R(3) = \{y \in B \mid 3Ry\} = \emptyset$.

(i) T F If $r, s, t$ are logical statements, and $r$ is true, $s$ is false, and $t$ is true, then $(\sim r) \lor (s \land t)$ is ...
Solution: F: $(\sim r) \lor (s \land t) = (\sim T) \lor (F \land T) = T \lor F = T$.

(j) T F If $D$ and $E$ are any sets, then $D \cup E \subseteq D$.
Solution: F: $D \cup E \supseteq D$. 

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