

Discrete Structures, Test 2  
 Monday, November 13, 2006  
 SOLUTIONS

1. (12 pts) Short answer. Put your answer in the box. No partial credit.

(a) Consider  $x = 1100$  and  $y = 0101$  to be elements of the Boolean algebra  $B_4$ . Compute  $y' \wedge x$ .

**Solution:**  $1000$ .  $y' \wedge x = (0101)' \wedge 1100 = 1010 \wedge 1100 = 1000$ .

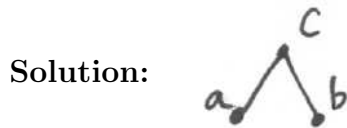
(b) How many elements does the Boolean algebra  $B_6$  have?

**Solution:**  $2^6 = 64$ .

(c) Compute  $\lfloor -7.94 \rfloor$ .

**Solution:**  $-8$ .

(d) Consider the relation  $R$  on  $\{a, b, c\}$  given by  $R = \{(a, a), (a, c), (b, b), (b, c), (c, c)\}$ . If  $R$  is a partial order, draw its Hasse diagram in the box. If it is not a partial order, write “not a partial order” in the box.

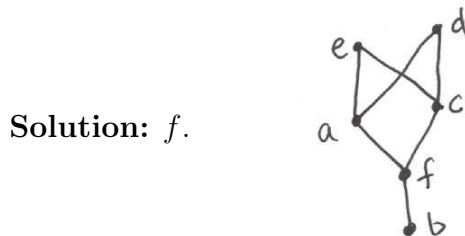


(e) Consider the relation  $S$  on  $\{1, 2, 3, 4\}$  whose matrix is given by  $\mathbf{M}_S = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ .

Let  $\mathbf{W}_0 = \mathbf{M}_S$ . Compute the next matrix  $\mathbf{W}_1$  under Warshall's algorithm.

**Solution:**  $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$  In going from  $\mathbf{W}_0$  to  $\mathbf{W}_1$ ,  $k = 1$ . The first row of  $\mathbf{W}_0$  has 1s in columns  $j = 3, 4$ , while the first column of  $\mathbf{W}_0$  has a 1 in row  $i = 2$ . So we add ones to  $\mathbf{W}_0$  in places  $(i, j) = (2, 3), (2, 4)$  to form  $\mathbf{W}_1$ .

(f) Given the Hasse diagram below, compute the greatest lower bound  $\text{GLB}(a, c) = a \wedge c$ . (Or state that no greatest lower bound exists.)

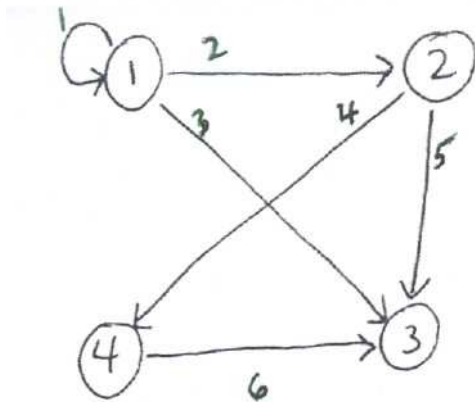


2. (4 pts) Consider the partition  $\mathcal{P} = \{\{A, C, E\}, \{B\}, \{D\}\}$  of the set  $\{A, B, C, D, E\}$ . Draw the matrix of the equivalence relation corresponding to  $\mathcal{P}$ .

**Solution:** 
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

3. (6 pts) Consider the relation given by the following digraph. Label the edges of the digraph, and write down the arrays HEAD, TAIL, VERT, NEXT, corresponding to this labelling.

**Solution:** Label as follows:



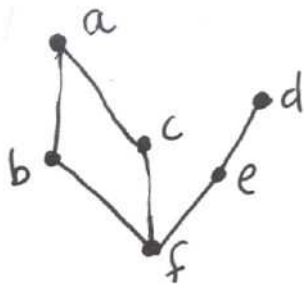
HEAD [1,2,3,4,3,3]

TAIL [1,1,1,2,2,4]

VERT [1,4,0,6]

NEXT [2,3,0,5,0,0]

4. (8 pts) Consider the Hasse diagram for the poset  $(G, \leq)$  below.



- (a) Write down all the incomparable pairs in  $(G, \leq)$ .

**Solution:**  $a, d$   $a, e$   $b, c$   $b, d$   $b, e$   $c, d$   $c, e$ .

(b) List all maximal elements of  $(G, \leq)$ .

**Solution:**  $a, d$ .

(c) Is there a least element of  $(G, \leq)$ ? Write it down or explain why there is no least element.

**Solution:**  $f$ .

(d) List all upper bounds of the set  $H = \{a, e, f\} \subset G$ . (Or write “none” if there are none.)

**Solution:** none. (There is no single element of  $G$  which is  $\geq$  both  $a$  and  $e$ .)

Consider the relation  $R$  on the set  $\{1, 2, 3, 4, 5\}$  with matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. (24 pts) True/False. Circle T or F. No explanation needed. For (a)-(d), refer to relation  $R$  whose matrix  $\mathbf{M}_R$  is above.

(a) T F  $R$  is reflexive.

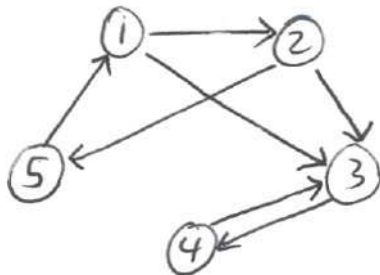
**Solution: F.** For example  $1 \not R 1$ . (There are not all 1s on the diagonal.)

(b) T F  $R$  is irreflexive.

**Solution: T.** There are all 0s on the diagonal.

(c) T F  $4R^31$ .

**Solution: F.** The easiest way to proceed is to write down the digraph of  $R$ .



Then we see that there is no path of length 3 from 4 to 1. Alternately, we can compute  $\mathbf{M}_{R^3} = \mathbf{M}_R \odot \mathbf{M}_R \odot \mathbf{M}_R$ , which is

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(d) T F  $R$  is a partial order.

**Solution: F.**  $R$  is not even reflexive.

(e) T F A one-to-one, everywhere-defined function  $f : Z \rightarrow Z$  must be onto ( $Z$  is as usual the set of integers).

**Solution: F.**  $f(n) = 2n$  is from  $Z \rightarrow Z$ , is one-to-one and is everywhere defined, but is not onto (no odd numbers are in the range of  $f$ ).

(f) T F  $\Theta(2^n + n^{1000}) = \Theta(2^n)$ .

**Solution: T.**  $\Theta(2^n) > \Theta(n^{1000})$  since exponential functions grow faster than any power of  $n$ . The sum of 2 functions of unequal  $\Theta$  class has  $\Theta$  class of the faster growing one ( $2^n$  in this case).

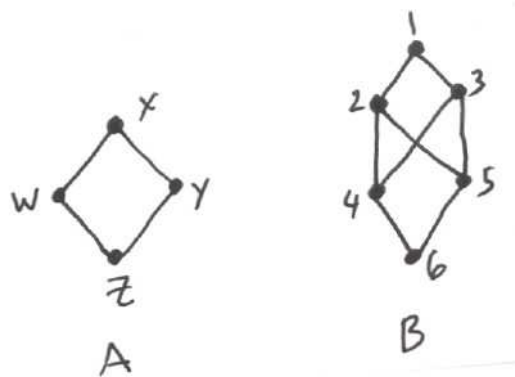
(g) T F Every finite poset has at least one maximal element.

**Solution: T.** We discussed this theorem in class.

(h) T F  $(D_{60}, |)$  is a Boolean algebra. (Recall  $D_{60}$  is the set of all divisors of 60.)

**Solution: F.**  $60 = 2^2 \cdot 3 \cdot 5$  is not the product of *distinct* primes.

For (i) and (j), consider the Hasse diagrams for posets  $(A, \leq)$  and  $(B, \leq)$  below.



- (i) T F  $(A, \leq)$  is a lattice.  
**Solution: T.** In fact it is isomorphic to the Boolean algebra  $B_2$ .
- (j) T F  $(B, \leq)$  is a lattice.  
**Solution: F.** LUB $(4, 5)$  does not exist (the upper bounds of  $\{4, 5\}$  are  $1, 2, 3$ , and there is no *least* upper bound among them).
- (k) T F If  $S$  and  $T$  are relations on a set  $C$ , then the matrix  $\mathbf{M}_{S \circ T}$  of the composition relation  $S \circ T$  satisfies  $\mathbf{M}_{S \circ T} = \mathbf{M}_S \odot \mathbf{M}_T$ .  
**Solution: F.**  $\mathbf{M}_{S \circ T} = \mathbf{M}_T \odot \mathbf{M}_S$ .
- (l) T F Any antisymmetric relation is irreflexive.  
**Solution: F.** Any *asymmetric* relation is irreflexive. On the other hand, any partial order (such as  $\leq$ ) is antisymmetric and reflexive (and so is not irreflexive).