Discrete Structures, Test 1
SOLUTIONS

1. (12 pts) Short answer. Put your answer in the box. No partial credit.

(a) Compute \[ \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \land \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \]

Solution: \[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \]

(b) Compute the combination \( 6 \choose 3 \). Your answer should be in the form of an integer.

Solution: \( 6 \choose 3 = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20. \)

(c) What is the Boolean matrix for the relation whose digraph is below?

Solution: \[ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}. \]

(d) If \( A = \{1, 2\} \) and \( B = \{2, 6, 8\} \), compute the symmetric difference \( A \oplus B \).

Solution: \( A \oplus B = (A \cup B) \setminus (A \cap B) = \{1, 2, 6, 8\} \setminus \{2\} = \{1, 6, 8\}. \)

(e) Write \( 103_4 \) as a base-10 integer.

Solution: \( 103_4 = 1 \cdot 4^2 + 0 \cdot 4^1 + 3 \cdot 4^0 = 16 + 0 + 3 = 19. \)

(f) Consider a sequence defined recursively by

\[ f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \text{for} \ n \geq 3. \]

What is \( f_6 \)? Your answer should be in the form of an integer.

Solution: \( f_3 = f_2 + f_1 = 1 + 1 = 2, \ f_4 = f_3 + f_2 = 2 + 1 = 3, \ f_5 = f_4 + f_3 = 3 + 2 = 5, \ f_6 = f_5 + f_4 = 5 + 3 = 8. \)

2. (12 pts) Consider the matrix \( A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \).

(a) Compute the transpose matrix \( A^T \).

Solution: \( A^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \)
(b) If $A = M_R$ for a relation $R$ on $\{1, 2, 3, 4\}$, draw the digraph of $R$.

(c) For the relation $R$ defined in the previous part, write down a cycle in $R$ of length 3.

Solution: 2, 1, 4, 2 is an example.

3. (6 pts) Prove the following statement by mathematical induction: For each $n \geq 1$,

$$P(n) : \quad n < 2^n.$$  

Solution: Basis step: $n = 1$. $1 < 2^1 = 2$ is true, so the basis step is checked.
Induction step: For $k \geq 1$, assume $P(k)$ to try to prove $P(k+1)$: So we assume $k < 2^k$, and we want to prove $k + 1 < 2^{k+1}$. Compute

$$2^{k+1} = 2 \cdot 2^k = (1 + 1)2^k = 2^k + 2^k > k + 2^k > k + 1,$$

where the first $>$ is by the inductive hypothesis $P(k)$, and the second $>$ is since $2^k > 1$ for all $k \geq 1$. So the induction step holds and we have proved the statement for all $n \geq 1$.

4. (6 pts) A fair 6-sided die is rolled twice. What is the probability that the sum of the two numbers rolled is exactly 9? Show your work.

Solution: The sample space $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ and $|S| = 6 \times 6 = 36$. The event space $E$ is the all the possible pairs whose sum is 9: so $E = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$ and $|E| = 4$. So the probability is 

$$\frac{|E|}{|S|} = \frac{4}{36} = \frac{1}{9}.$$

5. (4 pts) Consider the mathematical structure $(\mathbb{Z}^+, +, *, \div)$, where $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$ is the set of positive integers. Is $\mathbb{Z}^+$ closed under the binary operation $\div$? Explain your answer.

Solution: $\mathbb{Z}^+$ is not closed under $\div$, since for example $4 \div 5 = 0.8 \notin \mathbb{Z}^+$.

6. (20 pts) True/False. Circle T or F. No explanation needed.
For questions (a) and (b) below, consider the following digraph of a relation $R$ on the set $\{A, B, C, D\}$:
(a)  T  F  $C R^* C$.
Solution:  T: Recall $C R^* C$ if $C = C$ or $C R^\infty C$. Since $C = C$, $C R^* C$ is true.

(b)  T  F  $A R^\infty C$.
Solution:  T: $A R B$ and $B R C$.

(c)  T  F  If $p, q, r$ are all true logical statements then $(p \land q) \implies (\sim r)$ is ...
Solution:  F: $(p \land q) \implies (\sim r)$ is $(T \land T) \implies (\sim T)$ is $T \implies F$ is $F$.

(d)  T  F  $4P_4 = 24$.
Solution:  T: $4P_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

(e)  T  F  The Boolean product $\odot$ is commutative on $2 \times 2$ Boolean matrices.
Solution:  F: For example, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(f)  T  F  The meet operator $\land$ is commutative on $2 \times 2$ Boolean matrices.
Solution:  T: This is true since $\land$ is commutative on bits, and
$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \land \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \land b_{11} & a_{12} \land b_{12} \\ a_{21} \land b_{21} & a_{22} \land b_{22} \end{bmatrix}$. 

3
(g) T F Any prime number $p$ is a factor of $(p - 1)!$

Solution: F: In fact no prime number $p$ is a factor of $(p - 1)!$. This is because of the unique factorization theorem for positive integers: Each integer $k$ between 1 and $p - 1$ can be written uniquely as a product of prime factors:

$$k = \prod_{i=1}^{s_k} q_{i,k}^{r_{i,k}}$$

where each $q_{i,k}$ is a prime which must be $< p$ since $k < p$. Thus

$$(p - 1)! = \prod_{k=1}^{p-1} k = \prod_{k=1}^{p-1} \prod_{i=1}^{s_k} q_{i,k}^{r_{i,k}}$$

is the prime decomposition of $(p - 1)!$, which does not involve $p$. Since $p$ is prime, then $p$ does not divide into $(p - 1)!$

(h) T F If $A$ and $B$ are sets, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: T: This DeMorgan’s Law.

(i) T F The number of possible passwords consisting of 5 lower-case letters is $26^5$.

Solution: F: The number of possible passwords is $26^5$.

(j) T F $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity element for the binary operation of matrix addition of $3 \times 3$ matrices.

Solution: F: This is the identity element for matrix multiplication, not matrix addition. For example

$$\begin{bmatrix} 0 & 1 & 1 \\ 7 & 0 & 1 \\ 8 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 & 1 \\ 7 & 0 & 1 \\ 8 & 1 & 5 \end{bmatrix}.$$