Discrete Structures, Test 1
SOLUTIONS

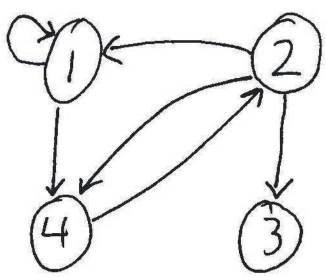
1. (12 pts) Short answer. Put your answer in the box. No partial credit.

   (a) Compute $\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$.

   Solution: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$.

   (b) Compute the combination ${}_6C_3$. Your answer should be in the form of an integer.
   Solution: ${}_6C_3 = \frac{6\cdot5\cdot4}{3\cdot2\cdot1} = 20$.

   (c) What is the Boolean matrix for the relation whose digraph is below?



   Solution: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

   (d) If $A = \{1, 2\}$ and $B = \{2, 6, 8\}$, compute the symmetric difference $A \oplus B$.
   Solution: $A \oplus B = (A \cup B) \setminus (A \cap B) = \{1, 2, 6, 8\} \setminus \{2\} = \{1, 6, 8\}$.

   (e) Write $(103)_4$ as a base-10 integer.
   Solution: $(103)_4 = 1 \cdot 4^2 + 0 \cdot 4^1 + 3 \cdot 4^0 = 16 + 0 + 3 = 19$.

   (f) Consider a sequence defined recursively by

   $$f_1 = 1, \qquad f_2 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 3.$$

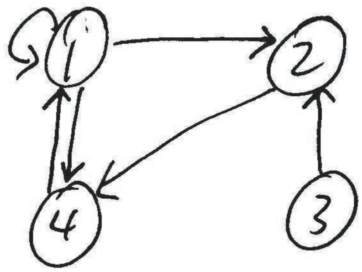
   What is f_6? Your answer should be in the form of an integer.
   Solution: $f_3 = f_2 + f_1 = 1 + 1 = 2$, $f_4 = f_3 + f_2 = 2 + 1 = 3$, $f_5 = f_4 + f_3 = 3 + 2 = 5$, $f_6 = f_5 + f_4 = 5 + 3 = 8$.

2. (12 pts) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

   (a) Compute the transpose matrix $\mathbf{A}^T$.

   Solution: $\mathbf{A}^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(b) If $\mathbf{A} = \mathbf{M}_R$ for a relation R on $\{1, 2, 3, 4\}$, draw the digraph of R.



(c) For the relation R defined in the previous part, write down a cycle in R of length 3.

Solution: $2, 1, 4, 2$ is an example.

3. (6 pts) Prove the following statement by mathematical induction: For each $n \geq 1$,

$$P(n): \qquad n < 2^n.$$

Solution: Basis step: $n = 1$. $1 < 2^1 = 2$ is true, so the basis step is checked.

Induction step: For $k \geq 1$, assume $P(k)$ to try to prove $P(k+1)$: So we assume $k < 2^k$, and we want to prove $k + 1 < 2^{k+1}$. Compute

$$2^{k+1} = 2 \cdot 2^k = (1+1)2^k = 2^k + 2^k > k + 2^k > k + 1,$$

where the first $>$ is by the inductive hypothesis $P(k)$, and the second $>$ is since $2^k > 1$ for all $k \geq 1$. So the induction step holds and we have proved the statement for all $n \geq 1$.

4. (6 pts) A fair 6-sided die is rolled twice. What is the probability that the sum of the two numbers rolled is exactly 9? Show your work.

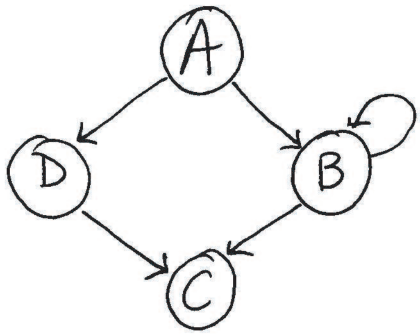
Solution: The sample space $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ and $|S| = 6 \times 6 = 36$. The event space E is the all the possible pairs whose sum is 9: so $E = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$ and $|E| = 4$. So the probability is

$$\frac{|E|}{|S|} = \frac{4}{36} = \frac{1}{9}.$$

5. (4 pts) Consider the mathematical structure $(Z^+, +, -, *, \div)$, where $Z^+ = \{1, 2, 3, 4, \ldots\}$ is the set of positive integers. Is Z^+ closed under the binary operation $\div$? Explain your answer.

Solution: Z^+ is not closed under $\div$, since for example $4 \div 5 = 0.8 \notin Z^+$.

6. (20 pts) True/False. Circle T or F. No explanation needed.
For questions (a) and (b) below, consider the following digraph of a relation R on the set $\{A, B, C, D\}$:



(a) T F $C\,R^*\,C$.
Solution: **T**: Recall $C\,R^*\,C$ if $C = C$ or $C\,R^\infty\,C$. Since $C = C$, $C\,R^*\,C$ is true.

(b) T F $A\,R^\infty\,C$.
Solution: **T**: $A\,R\,B$ and $B\,R\,C$.

(c) T F If p, q, r are all true logical statements then $(p \wedge q) \Longrightarrow (\sim r)$ is ...
Solution: **F**: $(p \wedge q) \Longrightarrow (\sim r)$ is $(T \wedge T) \Longrightarrow (\sim T)$ is $T \Longrightarrow F$ is F.

(d) T F ${}_4P_4 = 24$.
Solution: **T**: ${}_4P_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

(e) T F The Boolean product $\odot$ is commutative on 2×2 Boolean matrices.
Solution: **F**: For example, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(f) T F The meet operator $\wedge$ is commutative on 2×2 Boolean matrices.
Solution: **T**: This is true since $\wedge$ is commutative on bits, and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \wedge \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \wedge b_{11} & a_{12} \wedge b_{12} \\ a_{21} \wedge b_{21} & a_{22} \wedge b_{22} \end{bmatrix}.$$

(g) T F Any prime number p is a factor of $(p-1)!$
Solution: **F**: In fact no prime number p is a factor of $(p-1)!$. This is because of the unique factorization theorem for positive integers: Each integer k between 1 and $p-1$ can be written uniquely as a product of prime factors:

$$k = \prod_{i=1}^{s_k} q_{i,k}^{r_{i,k}},$$

where each $q_{i,k}$ is a prime which must be $< p$ since $k < p$. Thus

$$(p-1)! = \prod_{k=1}^{p-1} k = \prod_{k=1}^{p-1} \prod_{i=1}^{s_k} q_{i,k}^{r_{i,k}}$$

is the prime decomposition of $(p-1)!$, which does not involve p. Since p is prime, then p does not divide into $(p-1)!$

(h) T F If A and B are sets, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
Solution: **T**: This DeMorgan's Law.

(i) T F The number of possible passwords consisting of 5 lower-case letters is 5^{26}.
Solution: **F**: The number of possible passwords is 26^5.

(j) T F $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity element for the binary operation of matrix addition of 3×3 matrices.
Solution: **F**: This is the identity element for matrix multiplication, not matrix addition. For example

$$\begin{bmatrix} 0 & 1 & 1 \\ 7 & 0 & 1 \\ 8 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 & 1 \\ 7 & 0 & 1 \\ 8 & 1 & 5 \end{bmatrix}.$$