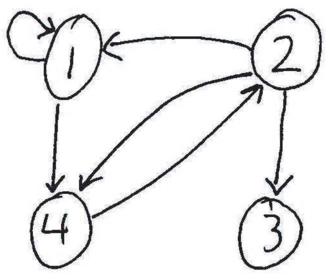
Discrete Structures, Test 2
SOLUTIONS

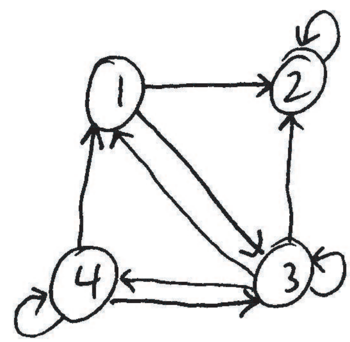
1. (12 pts) Short answer. Put your answer in the box. No partial credit.

   (a) If f is the mod-9 function, compute $f(57)$.
   Solution: $f(57) = 3$, since $57 = 6(9) + 3$.

   (b) Given the digraph of the relation R on $\{1, 2, 3, 4\}$ below, draw the digraph of the complementary relation $\bar{R}$ in the box.



Solution:



   (c) If $A = \{2, 3, 4, 5, 6\}$, list the minimal elements of A with respect to the partial order of divisibility $|$.

   Solution: $a = 2, 3, 5$ are minimal, since for each of $a = 2, 3, 5$, none of the other elements of A divides into a.

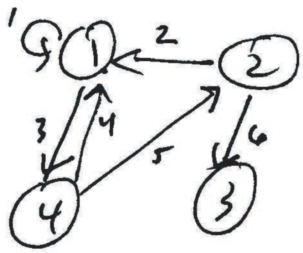
   (d) According to DeMorgan's law, what an equivalent Boolean expression to $(x \wedge y)'$?
   Solution: $x' \vee y'$.

   (e) What is the Θ class of $g(n) = 2n^2 + \sqrt{n} - n \lg n$? Your choices are $\Theta(\lg n)$, $\Theta(\sqrt{n})$, $\Theta(n \lg n)$, $\Theta(n^2)$, or $\Theta(2^n)$.
   Solution: $\Theta(n^2)$. This is because $g(n)$ is the sum of 3 functions $2n^2$, $\sqrt{n}$ and $-n \lg n$. $2n^2$ is $\Theta(n^2)$ has higher order growth than $\sqrt{n} = n^{\frac{1}{2}}$, since $2 > \frac{1}{2}$. Also $2n^2$ has higher order growth than $-n \lg n$, since $-n \lg n$ is $\Theta(n \lg n)$ and $n \lg n = n \cdot \lg n$, while $n^2 = n \cdot n$, and so n^2 has higher order growth since n has higher order growth than $\lg n$.

   (f) Consider $B = \{a, b, c\}$ and the relation R on B given by $\{(a, b), (a, c), (b, b)\}$. What is the reflexive closure of R?
   Solution: $\{(a, a), (a, b), (a, c), (b, b), (c, c)\}$.

2. (8 pts) Consider the relation R on $\{1, 2, 3, 4\}$ with matrix $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

   (a) Draw the digraph of R, and label the edges from 1 to 6.
   Solution:



(b) According to your labeling of the edges above, write down the arrays HEAD, TAIL, VERT, and NEXT which would appear in a computer representation of this digraph.

Solution:

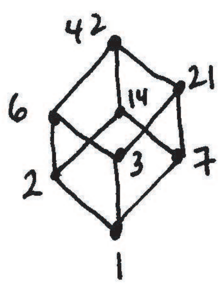
TAIL	$[1, 2, 1, 4, 4, 2]$
HEAD	$[1, 1, 4, 1, 2, 3]$
VERT	$[1, 2, 0, 4]$
NEXT	$[3, 6, 0, 5, 0, 0]$

3. (4 pts) Consider the linear order on $\{!, @, \#, \$\}$ given by $! < @ < \# < \$$. Put the four strings !@$, !!!, #$@, #@$ in lexigraphic order from smallest to largest.

Solution: $!!! \prec !@\$ \prec \#@\$ \prec \#\$@$.

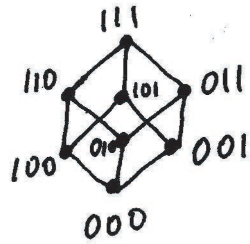
4. (8 pts)

(a) Draw the Hasse diagram of the lattice $(D_{42}, |)$.



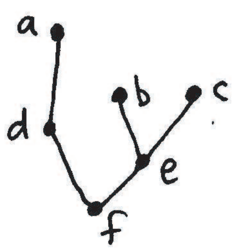
(b) Show $(D_{42}, |)$ is a Boolean algebra by exhibiting an isomorphism f from $(D_{42}, |)$ to B_n for some n. What is n? (A nice picture of the Hasse diagrams will suffice to exhibit the isomorphism.)

Solution: $n = 3$, as can be seen by comparing the Hasse diagram above with the Hasse diagram for B_3 below:



(c) In the lattice $(D_{42}, |)$, what is the complement of 21?

Solution: The complement of 21 is 2, since $2 \vee 21 = \text{LCM}(2, 21) = 42$ and $2 \wedge 21 = \text{GCD}(2, 21) = 1$. Alternately, you can compare to the Hasse diagram of B_3 above: the isomorphism takes 21 to 011, whose dual in B_3 is 100. On the other hand, the inverse of the isomorphism takes 100 back to 2, and so the complement of 21 in D_{42} is 2.



5. (8 pts) Consider the poset whose Hasse diagram is above.

(a) List all the maximal elements of the poset above.

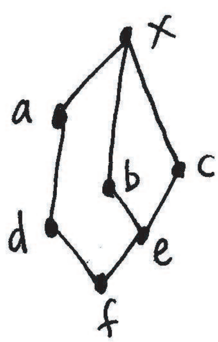
Solution: a, b, c.

(b) List all the minimal elements of the poset above.

Solution: f.

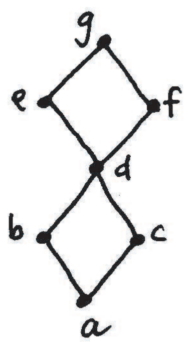
(c) By adding a single element x to the Hasse diagram above, together with appropriate lines from x, make a new Hasse diagram whose greatest element is x. Draw the new Hasse diagram below. Is the resulting poset a lattice?

Solution:



This poset is a lattice. You need only check the incomparable pairs $(a, b), (a, c), (b, c), (d, b), (d, c)$ to see that each pair has a GLB and LUB.

6. (20 pts) True/False. Circle T or F. No explanation needed.
For questions (a) through (e) below, consider the poset $(A, \leq)$ given by the following Hasse diagram.



(a) T F $e = f$.
Solution: F: They are represented by different dots on the Hasse diagram: So even though they're incomparable, they're not the same element.

(b) T F a is the least element of $(A, \leq)$.
Solution: T: $a <$ all the other elements of the poset.

(c) T F $\leq$ is a linear order on A.
Solution: F: There are incomparable pairs, e.g. e, f.

(d) T F $(A, \leq)$ is a lattice.
Solution: T: Check the incomparable pairs: $\text{LUB}(e, f) = g$, $\text{GLB}(e, f) = d$, $\text{LUB}(b, c) = d$, $\text{GLB}(b, c) = a$.

(e) T F $(A, \leq)$ is a Boolean algebra.
Solution: F: $|A| = 7$, which is not a power of 2.

(f) T F If R and S are relations on a set B, and $R \circ S$ is the composition of S and R, then their matrices satisfy $\mathbf{M}_{R \circ S} = \mathbf{M}_R \odot \mathbf{M}_S$.
Solution: F: $\mathbf{M}_{R \circ S} = \mathbf{M}_S \odot \mathbf{M}_R$.

(g) T F 3^n is $O(2^{2n})$.
Solution: T: $2^{2n} = (2^2)^n = 4^n$ and 3^n is $O(4^n)$ since $3 < 4$.

(h) T F The transitive closure of a symmetric relation on a set C is symmetric.
Solution: T: The transitive closure is formed by adding to the relation all those pairs (a, b) for which there is a path from a to b. Since the original relation is symmetric, we can follow the path backward to find a path from b to a also, and so (b, a) is in the transitive closure as well.

(i) T F Define a function $f\colon Z \to Z^+$ from the integers to the positive integers by

$$f(n) = \begin{cases} 2n+2 & \text{for} \quad n \geq 0, \\ 2|n|-1 & \text{for} \quad n < 0. \end{cases}$$

Then f is a one-to-one correspondence. (Hint: Compute $f(-2), f(-1), f(0), f(1), f(2)$, etc.)
Solution: T: We can find a formula for the inverse function $f^{-1}\colon Z^+ \to Z$:

$$f^{-1}(m) = \begin{cases} (m-2)/2 & \text{for} \quad m \text{ even}, \\ -(m+1)/2 & \text{for} \quad m \text{ odd}. \end{cases}$$

(j) T F If $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then the relation R is an equivalence relation.
Solution: T: R corresponds to the partition $\{\{1,3\},\{2\}\}$.