Discrete Structures: Sample Questions, Exam 1, Solutions

1. Prove by mathematical induction that $3 \mid (n^3 - n)$ for every positive integer $n$.

**Answer:** Basis step $n = 1$: $n^3 - n = 1^3 - 1 = 0$, and so we must check that $3 \mid 0$. This is true since $0 = 3(0)$.

Induction step. For $k \geq 1$, assume $P(k)$, which is that $3 \mid (k^3 - k)$. We want to show $P(k+1)$. In other words, we want to show that $3 \mid [(k+1)^3 - (k+1)]$.

So by $P(k)$, we know that there is an integer $s$ satisfying

$$P(k) : \quad k^3 - k = 3s.$$ 

To show $P(k+1)$, compute

$$[(k+1)^3 - (k+1)] = (k^3 + 3k^2 + 3k + 1) - (k + 1)$$
$$= k^3 + 3k^2 + 2k$$
$$= (k^3 - k) + k + (3k^2 + 2k)$$
$$= 3s + 3(k^2 + k) \quad \text{by } P(k)$$
$$= 3(s + k^2 + k).$$

Therefore $3 \mid [(k+1)^3 - (k+1)]$, and we have shown $P(k+1)$.

Note to get from the second line to the third in the computation, we want to use $P(k)$. So therefore, in the second line, we want to find the expression $k^3 - k$. This is the motivation in going from the second to the third lines.

Since we have the basis step $n = 1$ and the induction step, the proof is complete.

2. Write down a formula for the sequence

$3, 4, 6, 9, 13, 18, 24, 31, 38, \ldots$

Is your formula recursive or explicit?

**Answer:** $a_1 = 3$. $a_n = a_{n-1} + (n - 1)$. This is a recursive formula.
3. Consider the matrices

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}.
\]

Compute \(A \odot B\), \(AB\), \(B \odot A\) and \(A \wedge B\).

Assume \(A\) is the matrix of a relation. Draw the corresponding digraph.

Answer:

\[
A \odot B = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

\[
AB = \begin{bmatrix}
1 & 1 & 2 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

\[
B \odot A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

\[
A \wedge B = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

For the digraph, see the picture file.

4. Use the Euclidean algorithm to compute the greatest common divisor \(GCD(75, 12)\). Show your work. Compute the least common multiple \(LCM(75, 12)\).

Answer:

\[
75 = 6(12) + 3
\]

\[
12 = 4(3) + 0
\]

So the last one before the 0 is 3. So \(GCD(75, 12) = 3\).

\[
LCM(75, 12) = \frac{(75)(12)}{GCD(75, 12)} = \frac{(75)(12)}{3} = (75)4 = 300.
\]
5. How many ways can a committee of 3 faculty members and 2 students be selected from 7 faculty members and 8 students? Show your work.

**Answer:** Task $T_1$: to choose 3 faculty members from 7, there are 
\[
\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35
\]
ways. Task $T_2$: to choose 2 students from 8, there are 
\[
\binom{8}{2} = \frac{8 \cdot 7}{2 \cdot 1} = 28
\]
ways. All together, there are $35 \cdot 28 = 980$ ways of choosing this committee.

6. Two fair six-sided dice are rolled and the sum $s$ of the numbers coming up is recorded. What is the probability that $s \geq 10$? Show your work.

**Answer:** The sample space 
\[
A = \{(1,1), (1,2), \ldots, (6,6)\},
\]
and $|A| = 36$. The event that the sum is $\geq 10$ is 
\[
E = \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}.
\]
So the probability is 
\[
\frac{|E|}{|A|} = \frac{6}{36} = \frac{1}{6}.
\]
7. True/False. Circle T or F. No explanation needed.

(a) **T** F If $A$ and $B$ are any $2 \times 2$ matrices, then $AB = BA$.

**Answer:** F—Matrix multiplication is not in general commutative.

(b) **T** F If $A$, $B$ and $C$ are $2 \times 2$ matrices, then $(AB)C = A(BC)$.

**Answer:** T—Matrix multiplication is associative.

(i) **T** F Let $R = (A, \$)$ be the mathematical structure where $A$ is the set of even integers and $\$ is a unary operation on $A$ given by $\$a = \frac{a}{2}$ for every $a \in A$. $A$ is closed with respect to the operation $\$.

**Answer:** F—For $a = 2$, $\$a = \frac{1}{2} \notin A$.

(m) **T** F 49 and 77 are relatively prime.

**Answer:** F—GCD(49, 77) = 7 $\neq 1$.

(n) **T** F Let $A$ and $B$ be subsets of a universal set $U$. Then it is always true that $|A \cup B| = |A| + |B| - |A \cap B|$.

**Answer:** F—In general $|A \cup B| = |A| + |B| - |A \cap B|$, and so the equation is false if $A$ and $B$ are not disjoint.

(o) **T** F Two cards are dealt in succession from a standard shuffled 52-card deck. The number of possible 2-card hands is 1326.

**Answer:** T—The number of such hands is

$$52C_2 = \frac{52 \cdot 51}{2 \cdot 1} = 1326.$$ 

8. Let $a_n$ be the sequence recursively defined by $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1}a_{n-2}$ for $n \geq 3$. Compute the first five terms $a_1, \ldots, a_5$.

**Answer:**

$$
\begin{align*}
a_1 &= 2 \\
a_2 &= 3 \\
a_3 &= a_2a_1 = 2(3) = 6 \quad \text{(use } n = 3) \\
a_4 &= a_3a_2 = 6(3) = 18 \\
a_5 &= a_4a_3 = 18(6) = 108
\end{align*}
$$
9. A 6-sided die is rolled twice. What is the probability that the sum of the two rolls is exactly 8?

**Answer:** The sample space for two rolls of a die is

\[ A = \{(1,1), (1,2), \ldots, (6,5), (6,6)\}, \]

and \(|A| = 6^2 = 36\). The event given by the sum of the two rolls being 8 is given by

\[ E = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}, \]

and so \(|E| = 5\). So the probability is

\[ \frac{|E|}{|A|} = \frac{5}{36}. \]

10. Compute the truth table for the statement \([(p \land q) \lor r] \Rightarrow \lnot q\). Show your work.

**Answer:**

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11. Consider the relation \(R\) on \(A = \{1,2,3,4\}\) given by

\[ 1R2, \quad 2R3, \quad 3R3, \quad 3R4, \quad 4R3. \]

Draw the digraph of \(R\) and compute its connectivity relation \(R^\infty\). Draw the digraph of \(R^\infty\). (Hint: Try to determine \(R^\infty\) by inspection, not by computing with formulas.)

**Answer:** See picture page.