# Discrete Structures: Sample Questions, Exam 1, Solutions

1. Prove by mathematical induction that $3 \mid (n^3 - n)$ for every positive integer $n$.

   **Answer:** Basis step $n = 1$: $n^3 - n = 1^3 - 1 = 0$, and so we must check that $3 \mid 0$. This is true since $0 = 3(0)$.

   Induction step. For $k \geq 1$, assume $P(k)$, which is that $3 \mid (k^3 - k)$. We want to show $P(k+1)$. In other words, we want to show that $3 \mid [(k+1)^3 - (k+1)]$.

   So by $P(k)$, we know that there is an integer $s$ satisfying

   $$P(k): \qquad k^3 - k = 3s.$$

   To show $P(k+1)$, compute

   $$\begin{aligned} [(k+1)^3 - (k+1)] &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + k + (3k^2 + 2k) \\ &= 3s + 3(k^2 + k) \qquad \text{by } P(k) \\ &= 3(s + k^2 + k). \end{aligned}$$

   Therefore $3 \mid [(k+1)^3 - (k+1)]$, and we have shown $P(k+1)$.

   Note to get from the second line to the third in the computation, we want to use $P(k)$. So therefore, in the second line, we want to find the expression $k^3 - k$. This is the motivation in going from the second to the third lines.

   Since we have the basis step $n = 1$ and the induction step, the proof is complete.

2. Write down a formula for the sequence

   $$3, 4, 6, 9, 13, 18, 24, 31, 38, \ldots$$

   Is your formula recursive or explicit?

   **Answer:** $a_1 = 3$. $a_n = a_{n-1} + (n-1)$. This is a recursive formula.

3. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Compute $\mathbf{A} \odot \mathbf{B}$, $\mathbf{AB}$, $\mathbf{B} \odot \mathbf{A}$ and $\mathbf{A} \wedge \mathbf{B}$.

Assume $\mathbf{A}$ is the matrix of a relation. Draw the corresponding digraph.

**Answer:**

$$\begin{aligned} \mathbf{A} \odot \mathbf{B} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ \mathbf{AB} &= \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ \mathbf{B} \odot \mathbf{A} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ \mathbf{A} \wedge \mathbf{B} &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

For the digraph, see the picture file.

4. Use the Euclidean algorithm to compute the greatest common divisor GCD(75, 12). Show your work. Compute the least common multiple LCM(75, 12).

**Answer:**

$$\begin{aligned} 75 &= 6(12) + 3 \\ 12 &= 4(3) + 0 \end{aligned}$$

So the last one before the 0 is 3. So GCD(75, 12) = 3.

$$\text{LCM}(75, 12) = \frac{(75)(12)}{\text{GCD}(75, 12)} = \frac{(75)(12)}{3} = (75)4 = 300.$$

5. How many ways can a committee of 3 faculty members and 2 students be selected from 7 faculty members and 8 students? Show your work.

   **Answer:** Task $T_1$: to choose 3 faculty members from 7, there are

$$_7C_3 = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

   ways. Task $T_2$: to choose 2 students from 8, there are

$$_8C_2 = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

   ways. All together, there are $35 \cdot 28 = 980$ ways of choosing this committee.

6. Two fair six-sided dice are rolled and the sum $s$ of the numbers coming up is recorded. What is the probability that $s \geq 10$? Show your work.

   **Answer:** The sample space

$$A = \{(1,1), (1,2), \ldots, (6,6)\},$$

   and $|A| = 36$. The event that the sum is $\geq 10$ is

$$E = \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}.$$

   So the probability is

$$\frac{|E|}{|A|} = \frac{6}{36} = \frac{1}{6}.$$

7. True/False. Circle T or F. No explanation needed.

(a) T F If $\mathbf{A}$ and $\mathbf{B}$ are any $2 \times 2$ matrices, then $\mathbf{AB} = \mathbf{BA}$.
**Answer:** F—Matrix multiplication is not in general commutative.

(b) T F If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $2 \times 2$ matrices, then $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.
**Answer:** T—Matrix multiplication is associative.

(i) T F Let $R = (A, \$)$ be the mathematical structure where $A$ is the set of even integers and \$ is a unary operation on $A$ given by $\$a = \frac{a}{2}$ for every $a \in A$. $A$ is closed with respect to the operation \$.
**Answer:** F—For $a = 2$, $\$a = 1 \notin A$.

(m) T F 49 and 77 are relatively prime.
**Answer:** F—$\text{GCD}(49, 77) = 7 \neq 1$.

(n) T F Let $A$ and $B$ be subsets of a universal set $U$. Then it is always true that $|A \cup B| = |A| + |B|$.
**Answer:** F—In general $|A \cup B| = |A| + |B| - |A \cap B|$, and so the equation is false if $A$ and $B$ are not disjoint.

(o) T F Two cards are dealt in succession from a standard shuffled 52-card deck. The number of possible 2-card hands is 1326.
**Answer:** T—The number of such hands is

$$_{52}C_2 = \frac{52 \cdot 51}{2 \cdot 1} = 1326.$$

8. Let $a_n$ be the sequence recursively defined by $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1}a_{n-2}$ for $n \geq 3$. Compute the first five terms $a_1, \ldots, a_5$.

**Answer:**

$$\begin{aligned}
a_1 &= 2 \\
a_2 &= 3 \\
a_3 &= a_2 a_1 = 2(3) = 6 && \text{(use } n = 3\text{)} \\
a_4 &= a_3 a_2 = 6(3) = 18 \\
a_5 &= a_4 a_3 = 18(6) = 108
\end{aligned}$$

9. A 6-sided die is rolled twice. What is the probability that the sum of the two rolls is exactly 8?

**Answer:** The sample space for two rolls of a die is

$$A = \{(1,1), (1,2), \ldots, (6,5), (6,6)\},$$

and $|A| = 6^2 = 36$. The event given by the sum of the two rolls being 8 is given by

$$E = \{(2,6), (3,5), (4,4), (5,3), (6,2)\},$$

and so $|E| = 5$. So the probability is

$$\frac{|E|}{|A|} = \frac{5}{36}.$$

10. Compute the truth table for the statement $[(p \wedge q) \vee r] \Rightarrow (\tilde{}q)$. Show your work.

**Answer:**

| $p$ | $q$ | $r$ | $p \wedge q$ | $(p \wedge q) \vee r$ | $\tilde{}q$ | $[(p \wedge q) \vee r] \Rightarrow (\tilde{}q)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $T$ | $T$ |

11. Consider the relation $R$ on $A = \{1, 2, 3, 4\}$ given by

$$1R2, \quad 2R3, \quad 3R3, \quad 3R4, \quad 4R3.$$

Draw the digraph of $R$ and compute its connectivity relation $R^\infty$. Draw the digraph of $R^\infty$. (Hint: Try to determine $R^\infty$ by inspection, not by computing with formulas.)

**Answer:** See picture page.