Discrete Structures, Test 1

Monday, February 20, 2006

Solutions

1. (12 pts) Short answer. Put your answer in the box. No partial credit.

   (a) Let $A = \{a, b, d, e\}$ and $B = \{a, c, f\}$. Compute the intersection $A \cap B$.
   Solution: $A \cap B = \{a\}$.

   (b) Compute $\mathbf{A} \odot \mathbf{B}$ for $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.
   Solution: $\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

   (c) For the matrix $\mathbf{A}$ above in part (b), compute the transpose matrix $\mathbf{A}^\top$.
   Solution: $\mathbf{A}^\top = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

   (d) Let R be the relation on the set $A = \{x, y, z\}$ given by
   $$R = \{(x, x), (x, z), (y, x), (z, y)\}.$$
   Compute the relative set $R(\{y, z\})$.
   Solution: $R(\{y, z\}) = \{x, y\}$.

   (e) Compute the greatest common divisor $\mathrm{GCD}(120, 84)$.
   Solution: Use the Euclidean algorithm and compute: $120 = 1(84) + 36$, $84 = 2(36) + 12$, $36 = 3(12) + 0$. So $\mathrm{GCD}(120, 84) = 12$. Alternately, compute $120 = 2^3 \cdot 3 \cdot 5$, $84 = 2^2 \cdot 3 \cdot 7$, and so $\mathrm{GCD}(120, 84) = 2^2 \cdot 3 = 12$.

   (f) Compute the combination ${}_5C_4$. Your answer should be an integer.
   Solution: Compute
   $${}_5C_4 = \frac{5!}{4!(5-4)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(1)} = 5.$$

2. (6 pts) Complete the following truth table: (Answers in bold)

p	q	$\sim p$	$p \vee q$	$(\sim p) \Rightarrow (p \vee q)$
T	T	**F**	**T**	**T**
T	F	**F**	**T**	**T**
F	T	**T**	**T**	**T**
F	F	**T**	**F**	**F**

3. (6 pts) Prove the following statement by mathematical induction: For each $n \geq 1$,

$$P(n): \qquad 1+2+3+\cdots+(n-1)+n = \frac{n(n+1)}{2}.$$

Solution: Basis step:

$$P(1): \qquad 1 = \frac{1(1+1)}{2}$$

is true.

Induction step: For $n \geq 1$, assume $P(n)$ is true:

$$1+2+3+\cdot+(n-1)+n = \frac{n(n+1)}{2}.$$

We want to show

$$P(n+1): \qquad 1+2+3+\cdots+(n-1)+n+(n+1) = \frac{(n+1)[(n+1)+1]}{2}.$$

So compute

$$\begin{aligned} 1+2+3+\cdots+(n-1)+n+(n+1) &= \frac{n(n+1)}{2}+(n+1) \qquad \text{by } P(n) \\ &= (n+1)\left(\frac{n}{2}+1\right) \\ &= (n+1)\left(\frac{n+2}{2}\right) \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)[(n+1)+1]}{2}. \end{aligned}$$

So $P(n+1)$ is verified, and this completes the induction step.

Thus $P(n)$ is true for all $n \geq 1$ by mathematical induction.

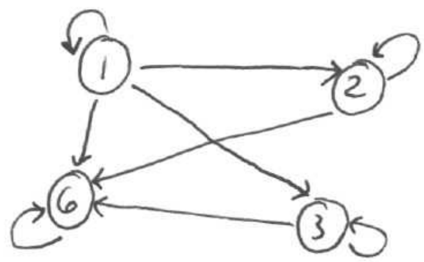

Figure 1:

4. (12 pts) Let $D = \{1, 2, 3, 6\}$ and consider the relation R on D given by aRb if and only if $a|\, b$. (Recall $a|\, b$ means a *divides* b, i.e. that $b \div a$ is an integer.)

   (a) Write down the relation R as a subset of $D \times D$. (In other words, write R as a set of ordered pairs in $D \times D$.)

   Solution:

   $$R = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\}.$$

   (b) Write down the matrix $\mathbf{M}_R$ associated to this relation R. (For ordering the rows and the columns, use the standard order $D = \{1, 2, 3, 6\}$.)

   Solution: $\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

   (c) Draw the digraph corresponding to R.

   Solution: See Figure 1

5. (4 pts) How many possible passwords can be formed from 5 lowercase letters? You do not have to simplify your answer.

   Solution: 26^5.

6. (20 pts) True/False. Circle T or F. No explanation needed. For (a) and (b), refer to the digraph of the relation R on $A = \{1, 2, 3, 4\}$ above.

(a) T F $3R^2 4$. (In other words, there is a path in R of length 2 from 3 to 4.)
Solution: T: $3R3$ and $3R4$.

(b) T F The domain $\text{Dom}(R)$ has exactly 3 elements.
Solution: F: $\text{Dom}(R) = \{1, 3\}$.

(c) T F 77 is prime.
Solution: F: $77 = 7(11)$.

(d) T F If $B = \{1, 2, 3, 4, 5\}$, then $\mathcal{P} = \{\{1, 5\}, \{2, 3\}, \{4\}\}$ is a partition of B.
Solution: T: $B = \{1, 5\} \cup \{2, 3\} \cup \{4\}$ and each element of B is contained in only one of these three sets.

(e) T F Two fair six-sided dice are rolled and the sum s is recorded. The probability that $s \leq 3$ is $1/12$.
Solution: T: The sample space $A = \{(1, 1), (1, 2), \ldots, (6, 6)\}$ has cardinality $|A| = 36$, and the event space $E = \{(1, 1), (1, 2), (2, 1)\}$. So the probability is

$$\frac{|E|}{|A|} = \frac{3}{36} = \frac{1}{12}.$$

(f) T F If $C = \{a, b, c, a, x\}$, then the cardinality $|C| = 5$.
Solution: F: $C = \{a, b, c, x\}$, and so $|C| = 4$.

(g) T F The matrix $\mathbf{N} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 3 & 0 \\ 7 & 0 & 0 \end{bmatrix}$ is a diagonal matrix.
Solution: F: To be a diagonal matrix, $\mathbf{N}$ would have to have its only nonzero elements on the main diagonal (not the other diagonal, as pictured here).

(h) T F The base 3 expansion of the decimal number 11 is $(102)_3$.
Solution: T: $(102)_3 = 1 \cdot 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0 = 9 + 0 + 2 = 11$.

(i) T F $\mathbb{R}$ the set of real numbers is closed under the binary operation $\div$ (division).
Solution: F: For example, $6 \div 0 \notin \mathbb{R}$.

(j) T F $120 = 5!$.
Solution: T: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.