1. (12 pts) Short answer. Put your answer in the box. No partial credit.

(a) Let $A = \{a, b, d, e\}$ and $B = \{a, c, f\}$. Compute the intersection $A \cap B$.

Solution: $A \cap B = \{a\}$.

(b) Compute $A \odot B$ for $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution: $A \odot B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(c) For the matrix $A$ above in part (b), compute the transpose matrix $A^T$.

Solution: $A^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(d) Let $R$ be the relation on the set $A = \{x, y, z\}$ given by

$R = \{(x, x), (x, z), (y, x), (z, y)\}$.

Compute the relative set $R(\{y, z\})$.

Solution: $R(\{y, z\}) = \{x, y\}$.

(e) Compute the greatest common divisor $\text{GCD}(120, 84)$.

Solution: Use the Euclidean algorithm and compute: $120 = 1(84) + 36$, $84 = 2(36) + 12$, $36 = 3(12) + 0$. So $\text{GCD}(120, 84) = 12$. Alternately, compute $120 = 2^3 \cdot 3 \cdot 5$, $84 = 2^2 \cdot 3 \cdot 7$, and so $\text{GCD}(120, 84) = 2^2 \cdot 3 = 12$.

(f) Compute the combination $5C_4$. Your answer should be an integer.

Solution: Compute

$$5C_4 = \frac{5!}{4!(5-4)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5.$$
2. (6 pts) Complete the following truth table: (Answers in bold)

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\sim p$</th>
<th>$p \lor q$</th>
<th>$(\sim p) \Rightarrow (p \lor q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$F$</td>
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</table>

3. (6 pts) Prove the following statement by mathematical induction: For each $n \geq 1$,

$$P(n) : \quad 1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n+1)}{2}.$$ 

**Solution:** Basis step:

$$P(1) : \quad 1 = \frac{1(1+1)}{2}$$

is true.

Induction step: For $n \geq 1$, assume $P(n)$ is true:

$$1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n+1)}{2}.$$ 

We want to show

$$P(n+1) : \quad 1 + 2 + 3 + \cdots + (n - 1) + n + (n+1) = \frac{(n+1)[(n+1)+1]}{2}.$$ 

So compute

$$1 + 2 + 3 + \cdots + (n - 1) + n + (n+1) = \frac{n(n+1)}{2} + (n+1) \quad \text{by } P(n)$$

$$= (n+1) \left( \frac{n}{2} + 1 \right)$$

$$= (n+1) \left( \frac{n+2}{2} \right)$$

$$= \frac{(n+1)(n+2)}{2}$$

$$= \frac{(n+1)[(n+1)+1]}{2}.$$ 

So $P(n+1)$ is verified, and this completes the induction step.

Thus $P(n)$ is true for all $n \geq 1$ by mathematical induction.
4. (12 pts) Let $D = \{1, 2, 3, 6\}$ and consider the relation $R$ on $D$ given by $aRb$ if and only if $a \mid b$. (Recall $a \mid b$ means $a$ divides $b$, i.e. that $b \div a$ is an integer.)

(a) Write down the relation $R$ as a subset of $D \times D$. (In other words, write $R$ as a set of ordered pairs in $D \times D$.)

**Solution:**

$$R = \{(1, 1), (1, 2), (1, 3), (1, 6), (2, 2), (2, 6), (3, 3), (3, 6), (6, 6)\}.$$

(b) Write down the matrix $M_R$ associated to this relation $R$. (For ordering the rows and the columns, use the standard order $D = \{1, 2, 3, 6\}$.)

**Solution:** $M_R = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}$.

(c) Draw the digraph corresponding to $R$.

**Solution:** See Figure 1

5. (4 pts) How many possible passwords can be formed from 5 lowercase letters? You do not have to simplify your answer.

**Solution:** $26^5$. 
6. (20 pts) True/False. Circle T or F. No explanation needed. For (a) and (b), refer to the digraph of the relation \( R \) on \( A = \{1, 2, 3, 4\} \) above.

(a) T F 3R24. (In other words, there is a path in \( R \) of length 2 from 3 to 4.)
Solution: T: 3R3 and 3R4.

(b) T F The domain \( \text{Dom}(R) \) has exactly 3 elements.
Solution: F: \( \text{Dom}(R) = \{1, 3\} \).

(c) T F 77 is prime.
Solution: F: 77 = 7(11).

(d) T F If \( B = \{1, 2, 3, 4, 5\} \), then \( \mathcal{P} = \{\{1, 5\}, \{2, 3\}, \{4\}\} \) is a partition of \( B \).
Solution: T: \( B = \{1, 5\} \cup \{2, 3\} \cup \{4\} \) and each element of \( B \) is contained in only one of these three sets.

(e) T F Two fair six-sided dice are rolled and the sum \( s \) is recorded. The probability that \( s \leq 3 \) is 1/12.
Solution: T: The sample space \( A = \{(1,1), (1,2), \ldots, (6,6)\} \) has cardinality \( |A| = 36 \), and the event space \( E = \{(1,1), (1,2), (2,1)\} \). So the probability is
\[
\frac{|E|}{|A|} = \frac{3}{36} = \frac{1}{12}.
\]

(f) T F If \( C = \{a, b, c, a, x\} \), then the cardinality \( |C| = 5 \).
Solution: F: \( C = \{a, b, c, x\} \), and so \( |C| = 4 \).

(g) T F The matrix \( N = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 3 & 0 \\ 7 & 0 & 0 \end{bmatrix} \) is a diagonal matrix.
Solution: F: To be a diagonal matrix, \( N \) would have to have its only nonzero elements on the main diagonal (not the other diagonal, as pictured here).

(h) T F The base 3 expansion of the decimal number 11 is \((102)_3\).
Solution: T: \((102)_3 = 1 \cdot 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0 = 9 + 0 + 2 = 11\).

(i) T F \( \mathbb{R} \) the set of real numbers is closed under the binary operation \( \div \) (division).
Solution: F: For example, \( 6 \div 0 \notin \mathbb{R} \).

(j) T F 120 = 5!. 
Solution: T: \( 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \).