Discrete Structures, Test 2, Solutions

1. (12 pts) Short answer. Put your answer in the box. No partial credit.

(a) Consider $x = 1101$ and $y = 0001$ to be elements of the Boolean algebra B_4. Compute $y' \wedge x$.

Solution: $y' \wedge x = (0001)' \wedge 1101 = 1110 \wedge 1101 = 1100$.

(b) Consider the relation R on $A = \{1, 2, 3\}$ whose matrix $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

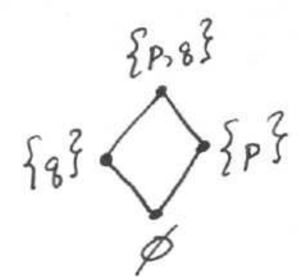
Compute the matrix $\mathbf{M}_{R^2}$.

Solution: $\mathbf{M}_{R^2} = (\mathbf{M}_R)^2_{\odot} = \mathbf{M}_R \odot \mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) For the relation R from part (b), compute the matrix of the reflexive closure of R.

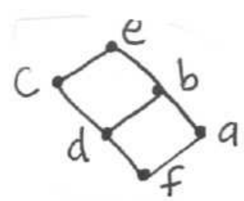
Solution: The reflexive closure of R is given by $R \cup \Delta$, and so its matrix is $\mathbf{M}_R \vee \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(d) Consider the set $B = \{p, q\}$. Draw the Hasse diagram of the poset $(P(B), \subseteq)$ consisting of subsets of B with the subset partial order.



(e) For the partial order whose Hasse diagram is below, compute the least upper bound $a \vee c$. (Write "does not exist" if there is no least upper bound.)

Solution: e

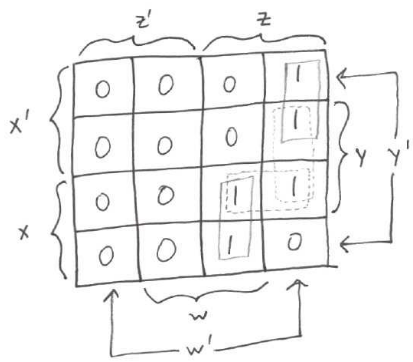


(f) If $b = 011 \in B_3$, compute the minterm E_b in terms of the Boolean variables x_1, x_2, x_3.

Solution: $x_1' \wedge x_2 \wedge x_3$.

2. (6 pts) Compute the Karnaugh map for the function f whose truth table is given to the right. Use the Karnaugh map to find a Boolean expression for the function f. Your answer should be as simple as possible.

Solution: The Karnaugh map is



x	y	z	w	$f(x,y,z,w)$
0	0	0	0	0
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	1
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1

The Karnaugh map should be read as follows: both the rectangles in solid lines should be included in your solution, and exactly one of the rectangles in the dotted line should be part of the solution (so include one or the other of the dotted rectangles, but not both).

Then either $f = (x \wedge z \wedge w) \vee (x' \wedge z \wedge w') \vee (x \wedge y \wedge z)$ or $f = (x \wedge z \wedge w') \vee (y \wedge z \wedge w') \vee (x \wedge y \wedge z)$ is acceptable (depending on the grouping in the Karnaugh map).

3. (6 pts) Arrange the following functions in order from lowest to highest order of growth (no explanation necessary):

$$2^n + n^{100}, \quad 2^{2n}, \quad \log_7 n, \quad \sqrt{3n}, \quad n^2 + \ln n, \quad 4, \quad n$$

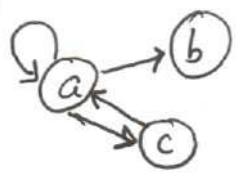
Solution: $\Theta(4) < \Theta(\log_7 n) < \Theta(\sqrt{3n}) < \Theta(n) < \Theta(n^2 + \ln n) < \Theta(2^n + n^{100}) < \Theta(2^{2n})$.

Why:

- 4 is a constant, and so it doesn't grow at all. It has the lowest Θ class.
- $\log_7 n = (\lg n)/(\lg 7)$, and so $\Theta(\log_7 n) = \Theta(\lg n)$.
- $\sqrt{3n} = \sqrt{3}\sqrt{n}$, and so $\Theta(\sqrt{3n}) = \Theta(n^{\frac{1}{2}})$ has higher order of growth than $\lg n$ since the exponent $\frac{1}{2} > 0$.
- $\Theta(n) = \Theta(n^1) > \Theta(n^{\frac{1}{2}}) = \Theta(\sqrt{3n})$.

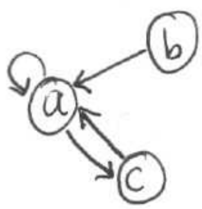
- $\Theta(n^2 + \ln n) = \Theta(n^2)$ since $\Theta(\ln n) < \Theta(n^2)$. Thus $\Theta(n^2 + \ln n) = \Theta(n^2) > \Theta(n^1) = \Theta(n)$.
- $\Theta(2^n + n^{100}) = \Theta(2^n)$ since $\Theta(n^{100}) < \Theta(2^n)$. Similarly, $\Theta(2^n) > \Theta(n^2) = \Theta(n^2 + \ln n)$.
- $2^{2n} = (2^2)^n = 4^n$, and so $\Theta(2^{2n}) = \Theta(4^n) > \Theta(2^n) = \Theta(2^n + n^{100})$.

4. (8 pts) Consider the relation R on $A = \{a, b, c\}$ given by the following digraph.



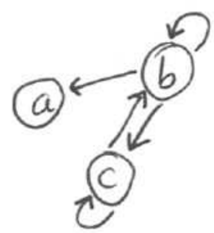
(a) Below, draw the digraph of the inverse relation R^{-1} of the relation R above.

Solution:



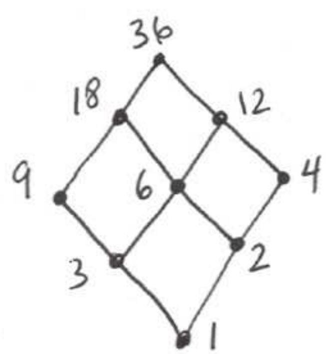
(b) Below, draw the digraph of the complementary relation $\overline{R}$ of the relation R above.

Solution:

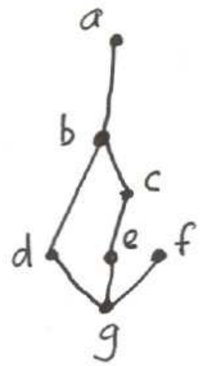


5. (4 pts) Draw the Hasse diagram of the lattice $(D_{36}, |)$ consisting of all the divisors of 36 with partial order of divisibility.

Solution:



Hasse diagram of $(A, \leq)$:



6. (20 pts) True/False. Circle T or F. No explanation needed. For (a) and (b), refer to the Hasse diagram above of the poset $(A, \leq)$.

(a) T F a is the only maximal element of the poset $(A, \leq)$.
Solution: F: f is also a maximal element.

(b) T F g is the least element of the poset $(A, \leq)$.
Solution: T.

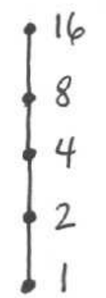
(c) T F Every partial order on a set B must be an antisymmetric relation on B.
Solution: T: Every partial order is reflexive, antisymmetric, and transitive.

(d) T F If $\leq$ is a linear order on a set C, then the lexigraphic order $\prec$ on $C \times C$ is also a linear order.
Solution: T: For example, dictionary order is a linear order (so one can look up words in a dictionary) because the order of the alphabet is a linear order.

(e) T F The number of steps needed to complete Warshall's algorithm for computing the transitive closure of a relation on a set with n elements, when thought of as a function of n, is $O(n^3)$.
Solution: T: Warshall's algorithm takes $O(n^3)$ steps, which is an improvement over the $O(n^4)$ steps taken by implementing the more obvious formula
$\mathbf{M}_{R^\infty} = \mathbf{M}_R \vee (\mathbf{M}_R)^2_\odot \vee \cdots \vee (\mathbf{M}_R)^n_\odot$.

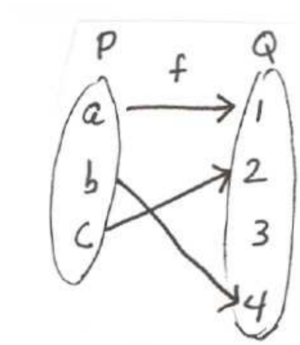
(f) T F If $x, y, z \in B_n$ the standard Boolean algebra with 2^n elements, then $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
Solution: T: $\vee$ distributes over $\wedge$.

(g) T F The partial order $|$ of divisibility is a linear order on D_{16} the set of divisors of 16.
Solution: T: The Hasse diagram of the poset $(D_{16}, |)$ is



(h) T F Every finite lattice has a least element.
Solution: T: If $L = \{a_1, \ldots, a_n\}$ is a finite lattice, its least element is given by $a_1 \wedge \cdots \wedge a_n$.

For (i) and (j), consider the relation f from $P = \{a, b, c\}$ to $Q = \{1, 2, 3, 4\}$ given by the following diagram:



(i) T F f is a function.
Solution: T: Each element in the domain P has only one range element: $f(a) = 1$, $f(b) = 4$, $f(c) = 2$.

(j) T F f is onto.
Solution: F. $3 \notin \text{Ran}(f)$.