1. Let $A = B = \{a, b, c\}$. Consider the relation $g = \{(a, b), (b, c), (c, c)\}$. Is $g$ one-to-one? Is $g$ onto? Why?

**Solution:** $g$ is not one-to-one, since for $c \in A$, $g(b) = g(c) = c$. $g$ is not onto, since $a \notin g(A)$.

2. Consider $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ defined by $f(a) = a^2$. Is $f$ one-to-one? Is $f$ onto? Why?

**Solution:** $f$ is one-to-one, since if $f(a) = f(b)$ for $a, b \in \mathbb{Z}^+$, then $a^2 = b^2$. Therefore, since elements of $\mathbb{Z}^+$ are positive, then we have $a = b$ and thus $f$ is one-to-one. $f$ is not onto. For example, 2 is not in the image $f(\mathbb{Z}^+)$.

3. Put the following functions in order from lowest to highest in terms of their $\Theta$ classes. (Some of the functions may be in the same $\Theta$ class. Indicate that on your list also.)

(a) $f_1(n) = n \log n$,
(b) $f_2(n) = n^\frac{3}{2}$,
(c) $f_3(n) = 10,000$,
(d) $f_4(n) = \sqrt{n}(n + \log n)$,
(e) $f_5(n) = 3^n$,
(f) $f_6(n) = 2^{n+2}$,
(g) $f_7(n) = 0.0001$.

**Solution:** These are listed from lowest to highest, with functions with the same $\Theta$ class listed on the same line:

- $f_3(n) = 10,000$, $f_7(n) = 0.0001$.
- $f_1(n) = n \log n$.
- $f_2(n) = n^\frac{3}{2}$, $f_4(n) = \sqrt{n}(n + \log n)$.
- $f_6(n) = 2^{n+2}$. 
• $f_5(n) = 3^n$.

Why? $f_3(n)$ and $f_7(n)$ are both constants, which is the lowest order of growth.

$f_1(n) = n \log n$, while $f_2(n) = n^{\frac{3}{2}}$. $f_1(n)$ has lower growth order since $f_1(n) = n \cdot \log n$ and $f_2(n) = n \cdot n^{\frac{1}{2}}$ and $\log n$ grows more slowly than $n^{\frac{1}{2}}$ (and indeed $\log n$ grows more slowly than any positive power of $n$).

$f_2(n)$ and $f_4(n)$ are in the same $\Theta$ class, since $n + \log n$ is in the $\Theta$ class of $n$. (This is since $\log n$ has slower order of growth than $n$.) Therefore, $f_4(n) = \sqrt{n}(n + \log n)$ has the same $\Theta$ class as $\sqrt{n} \cdot n = n^{\frac{3}{2}}$.

$f_6(n)$ and $f_5(n)$ are both of higher $\Theta$ class than $n^{\frac{3}{2}}$, since any exponential function with base $> 1$ grows faster than any power of $n$. $f_5(n) = 3^n$ has higher $\Theta$ class than $f_6(n) = 2^{n+2}$, since $f_6(n) = 4 \cdot 2^n$ has the same $\Theta$ class as $2^n$, and the $\Theta$ class of $3^n$ is higher than that of $2^n$ since the bases $3 > 2$.

4. Let $S = \{x, y, z\}$, and consider the set $P(S)$ with relation $R$ given by set inclusion. Is $R$ a partial order? Why or why not? (Carefully check the conditions needed for a relation to be a partial order.) Is $R$ a linear order? Again carefully check the conditions for $R$ to be a linear order.

**Solution:** $R$ is a partial order (i.e., $R$ is reflexive, antisymmetric, and transitive). Recall elements $A, B \in P(S)$ are subsets $A, B \subseteq S$. Then, $ARB$ if and only if $A \subseteq B$. $R$ is reflexive since $A \subseteq A$ always. $R$ is antisymmetric since if $A \subseteq B$ and $B \subseteq A$, then $A = B$. $R$ is transitive since if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

$R$ is not a linear order. In a linear order, each two elements are comparable. In particular $\{x\}, \{y\} \in P(S)$ are not comparable, so that $R$ is not a linear order.

5. Show that $(P(S), R)$ from the previous problem is isomorphic to the poset $D_{42}$ of divisors of 42 with relation given by divisibility.

**Solution 1:** Here is a quick proof: Since $S$ has 3 elements $(P(S), \subseteq)$ is isomorphic to the Boolean algebra $B_3$. Since $42 = 2 \cdot 3 \cdot 7$ is the product of 3 distinct primes, then $D_{42}$ is also isomorphic to $B_3$. It follows that $(P(S), \subseteq)$ and $D_{42}$ are isomorphic since the condition of 2 posets being isomorphic is an equivalence relation (and is thus transitive).
Solution 2: One can also draw the Hasse diagram.

An explicit one-to-one correspondence that preserves the partial orders is the following (there are other such correspondences as well):

\[
\begin{align*}
\{x, y, z\} &\leftrightarrow 42, \quad \{x, y\} \leftrightarrow 6, \quad \{x, z\} \leftrightarrow 14, \quad \{y, z\} \leftrightarrow 21, \\
\{x\} &\leftrightarrow 2, \quad \{y\} \leftrightarrow 3, \quad \{z\} \leftrightarrow 7, \quad \emptyset \leftrightarrow 1.
\end{align*}
\]

6. Consider the relation on \(A = \{a, b, c, d, e\}\).

\[
R = \{(a, a), (a, c), (b, c), (c, d), (e, a), (d, b)\}.
\]

Draw the corresponding digraph. Use Warshall’s algorithm to compute the matrix \(M_{R^\infty}\). Show all the intervening steps.

**Answer:**

Digraph:
Compute

\[ W_0 = M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ W_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \]

\[ W_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \]

\[ W_3 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \]

\[ W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \]

\[ M_{R^\infty} = W_5 = W_4. \]

7. Is there an infinite poset with a least element? Either write down an example or prove that this is impossible.

**Solution:** \((Z^+, \leq)\) is an infinite poset with least element 1.

8. Which of the following Hasse diagrams represent lattices?
Solution: (I), (III) and (IV) are lattices, for each pair of elements has a least upper bound and a greatest lower bound. (This fails in (II), since $a$ and $b$ do not have a least upper bound.)

9. For the examples in the previous problem, which represent finite Boolean algebras?

Solution: Only (I) represents a finite Boolean algebra. (It is isomorphic to $B_2$.) (II) does not since it is not even a lattice. (III) does not since it has $5 \neq 2^n$ elements. (IV) is not since there are no complements to each element: The unit element $I = f$ and the zero element $0 = i$ clearly. But given $h$, for example, there is no complement element $h'$ which satisfies $\text{GLB}(h, h') = 0$ and $\text{LUB}(h, h') = I$: Check each element: $\text{GLB}(f, h) = \text{GLB}(g, h) = \text{GLB}(h, h) = h \neq 0$, while $\text{LUB}(h, i) = h \neq I$.

10. Draw the Hasse diagram for the lattice $D_{18}$ consisting of the divisors of 18 with the partial order of divisibility.
11. Given the following Karnaugh map of a Boolean function, write the function as an equivalent Boolean polynomial.

\[
(y \land z) \lor (x \land y').
\]

See below for the corresponding rectangles:

12. Let \( x, y, z \) be Boolean variables. Use the rules of Boolean algebra (or a truth table with Karnaugh map) to simplify the following expression. Your answer should be as simple as possible.

\[
(x \land z) \lor (y' \lor (y' \land z)) \lor ((x \land y') \land z')
\]
Solution 1: Let $f = (x \land z) \lor (y' \lor (y' \land z)) \lor ((x \land y') \land z')$ and compute

\[
\begin{align*}
\text{f} & = (x \land z) \lor y' \lor (x \land y' \land z') & [\text{Simplify } y' \lor (y' \land z)] \\
& = y' \lor (x \land z) \lor (x \land y' \land z') & [\lor \text{ is commutative}] \\
& = y' \lor (x \land (z \lor (y' \land z')))) & [\land \text{ is distributive over } \lor] \\
& = y' \lor (x \land ((z \lor y') \land (z \lor z'))) & [\lor \text{ is distributive over } \land] \\
& = y' \lor (x \land ((z \lor y') \land I)) \\
& = y' \lor (x \land (z \lor y')) \\
& = (y' \lor x) \land (y' \lor (z \lor y')) & [\lor \text{ is distributive over } \land] \\
& = (y' \lor x) \land ((y' \lor y') \lor z) & [\lor \text{ is commutative and associative}] \\
& = (y' \lor x) \land (y' \lor z) \\
& = y' \lor (x \land z) & [\lor \text{ is distributive over } \land]
\end{align*}
\]

Solution 2: Again let $f = (x \land z) \lor (y' \lor (y' \land z)) \lor ((x \land y') \land z')$ and compute the truth table

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<th>$x$</th>
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<th>$x \land z$</th>
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From the associated Karnaugh map, we see that $f = y' \lor (x \land z)$. 
13. If $b = 1011 \in B_4$, write down the minterm $E_b$ in terms of the Boolean variables $x_1, x_2, x_3, x_4$.

Solution: $x_1 \land x_2' \land x_3 \land x_4$.

14. True/False. Circle T or F. No explanation needed.
(a) T F If $f$ is a one-to-one function from an infinite set $A$ to itself, then $f$ must be onto.
Solution: F. Problem 2 provides a counterexample.

(b) T F If $g$ is a one-to-one function from a finite set $A$ to itself, then $g$ must be onto.
Solution: T. Use Theorem 4 in Section 5.1.

(c) T F If $x$ is a Boolean variable, then $x \lor x = x$.
Solution: T.

(d) T F If $y$ is a Boolean variable, then $y \lor I = y$.
Solution: F. $y \lor I = I$.

(e) T F Let $B = \{0, 1\}$ with the standard partial order and let $A = B \times B \times B$ with the product partial order. Then $A$ is isomorphic as a lattice to $D_{60}$.
Solution: F. $A$ is a Boolean algebra, while $D_{60}$ is not (this is because $60 = 2^2 \cdot 3 \cdot 5$ has the prime 2 repeated in its prime decomposition).

(f) T F $D_{85}$ is a Boolean algebra.
Solution: T. $85 = 5 \cdot 17$ is a product of distinct primes.

(g) T F Every finite lattice has a least element.
Solution: T. If the elements of the lattice are $a_1, \ldots, a_n$, then $a_1 \land \cdots \land a_n$ is the least element.

(h) T F Every poset has a greatest element.
Solution: F. For example, the poset $(\mathbb{R}, \leq)$ has no greatest element; i.e., there is no largest real number.

(i) T F $f(n) = \log_5(n)$ is $O(\lg(n))$.
Solution: T. $\log_5(n) = \lg(n)/\lg(5)$.

(j) T F If $g$ is the mod-10 function, then $g(405) = 4$.
Solution: F. $g(405) = 5$.

(k) T F If $x$ and $y$ are Boolean variables, then $(x \land y)' = x' \lor y'$.
Solution: F. DeMorgan’s Laws state that $(x \land y)' = x' \lor y'$.

15. Consider the relation $R$ on $A = \{1, 2, 3, 4\}$ given by the matrix

$$M_R = \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$
Is $R$ a partial order? Why or why not?

**Solution:** $R$ is not a partial order since $R$ is not antisymmetric ($2R3$ and $3R2$ but $2 \neq 3$.)

16. Consider the poset $(A, \leq)$ given by the following Hasse diagram.

(a) List all the minimal elements of $(A, \leq)$.

**Solution:** $g$.

(b) List all the upper bounds of $B = \{c, d\} \subseteq A$.

**Solution:** $b, c$.

(c) List a pair of incomparable elements of $A$ (if such a pair exists).

**Solution:** $a, c$ and $d, e$ are incomparable pairs (you only need to write down one of the two pairs).

(d) Is $(A, \leq)$ a lattice? Why or why not?

**Solution:** $(A, \leq)$ is not a lattice since GLB($a, c$) does not exist ($d, e, f, g$ are all lower bounds, but there is no greatest element among them). Similarly, LUB($d, e$) does not exist.

17. Let $A = \{\ast, q, 1\}$, with partial order determined by $q < 1 < \ast$. Put the following elements of $A \times A \times A$ in lexicographic order:

$(q, q, 1), \ (q, 1, \ast), \ (\ast, *, q), \ (q, q, q), \ (\ast, 1, 1), \ (1, *, q), \ (q, 1, 1), \ (\ast, * , *)$

**Solution:**

$(q, q, q) \prec (q, q, 1) \prec (q, 1, 1) \prec (q, 1, \ast) \prec (1, *, q) \prec (\ast, 1, 1) \prec (\ast, *, q) \prec (\ast, *, *)$