Discrete Structures, Test 1

Monday, March 2, 2009

SOLUTIONS

1. (12 pts) Short answer. Put your answer in the box. No partial credit.

   (a) Compute $\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

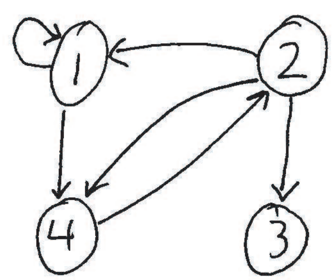
   Solution: $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

   (b) Compute the permutation ${}_6P_3$. Your answer should be in the form of an integer.

   Solution: ${}_6P_3 = 6(5)(4) = 120$.

   (c) What is the Boolean matrix for the relation whose digraph is below?

   Solution: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$



   (d) If $A = \{1, 2\}$ and $B = \{2, 6, 8\}$, compute the union $A \cup B$.

   Solution: $A \cup B = \{1, 2, 6, 8\}$.

   (e) Consider the relation R on $\{1, 2, 3\}$ whose matrix is $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Then the matrix of the complementary relation $\mathbf{M}_{\bar{R}}$ is

   Solution: $\mathbf{M}_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. The complementary relation $\bar{R}$ is defined to be $a\bar{R}b$ if and only if $a \not\!R b$. So the matrix has all the zeros switched to ones and vice versa.

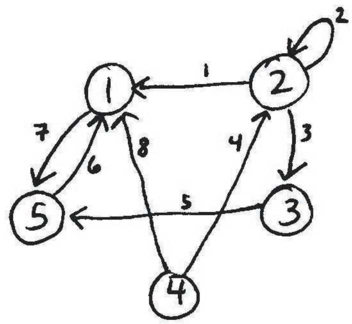
   (f) Let C and D be two sets with cardinalities $|C| = 4$ and $|D| = 5$. Assume that $|C \cap D| = 3$. Compute $|C \cup D|$.

   Solution: $|C \cup D| = |C| + |D| - |C \cap D| = 4 + 5 - 3 = 6$.

2. (12 pts) Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) If $\mathbf{A} = \mathbf{M}_R$ for a relation R on $\{1, 2, 3, 4, 5\}$, draw the digraph of R.

Solution:



(b) Number the edges of your digraph above, and then use this numbering to write down arrays TAIL, HEAD, VERT, NEXT, to represent the computer representation of the relation R.

Solution:
TAIL[2,2,2,4,3,5,1,4]
HEAD[1,2,3,2,5,1,5,1]
VERT[7,1,5,4,6]
NEXT[2,3,0,8,0,0,0,0]

(c) For the relation R defined in the previous part, write down two paths of length 3 from vertex 4 to vertex 5.

Solution: First path: $4R2, 2R1, 1R5$; Second path: $4R2, 2R3, 3R5$.

3. (5 pts) A fair coin is flipped three times. What is the probability that exactly two heads and one tail come up (any order of two heads and one tail is valid)? Show your work.

Solution: The probability is 3/8. The sample space, which is the space of 3 possible coin flips is

$$A = \{\text{heads}, \text{tails}\} \times \{\text{heads}, \text{tails}\} \times \{\text{heads}, \text{tails}\},$$

which has cardinality $|A| = 2^3 = 8$. On the other hand, the event space

$$E = \{(\text{heads}, \text{heads}, \text{tails}), (\text{heads}, \text{tails}, \text{heads}), (\text{tails}, \text{heads}, \text{heads})\}.$$

So the probability is $|E|/|A| = 3/8$.

4. (5 pts) Consider the sequence defined recursively by

$$\begin{cases} a_1 = 7 \\ a_n = a_{n-1} + 4 \quad \text{for } n \geq 2 \end{cases}$$

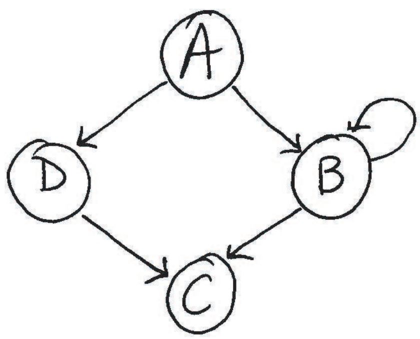
Use backtracking to find an explicit formula for the sequence a_n. Show your work.

Solution: Compute

$$\begin{aligned}
a_n &= a_{n-1} + 4 \\
&= (a_{n-2} + 4) + 4 \\
&= ((a_{n-3} + 4) + 4) + 4 \\
&= a_1 + 4 + \cdots + 4 \\
&= a_1 + (n-1)4 \\
&= 7 + 4(n-1) \\
&= 4n + 3
\end{aligned}$$

The key point is to realize that there are $n - 1$ fours in the sum on the fourth line.

5. (20 pts) True/False. Circle T or F. No explanation needed.
   For questions (a), (b) and (c) below, consider the following digraph of a relation R on the set $\{A, B, C, D\}$:



(a) T F R is transitive.
Solution: **F**. For example, ARD and DRC but $A \not R C$.

(b) T F R is antisymmetric.
Solution: **T**. $xRy \wedge yRx$ does imply $x = y$, since the only time the hypothesis happens is when $x = y = B$.

(c) T F AR^3C.
Solution: **T**. ARB, BRB, BRC forms a path of length 3 from A to C.

(d) T F Consider the mathematical structure $(Z, +, -, *, \div)$, where Z is the set of all integers. Then Z is closed under the binary subtraction operation $-$.
Solution: **T**. The difference of two integers is again an integer.

(e) T F Let p and q be two logical statements. Assume $p \implies q$ is false. Then p must be
Solution: **T**. The only way for $p \implies q$ to be false is if p is true and q is false.

(f) T F Let R be a relation on a set A. Then the symmetric closure of the reflexive closure of R is the same relation as the reflexive closure of the symmetric closure of R.
Solution: **T**. One way to see this is in terms of matrices. The matrix of the symmetric closure of R is $\mathbf{M}_R \vee \mathbf{M}_R^\top$, while the matrix of the reflexive closure of R is $\mathbf{M}_R \vee \mathbf{I}$. Then what we need to check is

$$(\mathbf{M}_R \vee \mathbf{M}_R^\top) \vee \mathbf{I} = (\mathbf{M}_R \vee \mathbf{I}) \vee (\mathbf{M}_R \vee \mathbf{I})^\top.$$

This works since $\vee$ is commutative and associative, $^\top$ distributes over $\vee$, $\mathbf{I}^\top = \mathbf{I}$, and $\mathbf{I} \vee \mathbf{I} = \mathbf{I}$.

(g) T F For k an integer, consider $P(k)$ to be the logical statement "k is even." Then $P(k) \implies P(k+1)$ is
Solution: **F**. For example, $P(2) \implies P(3)$ is false since $P(2)$ is true, while $P(3)$ is false.

(h) T F If A and B are subsets of a universal set, then $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
Solution: **F**. $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by DeMorgan's Law.

(i) T F The number of possible passwords consisting of 4 lowercase letters is ${}_{26}C_4$.
Solution: **F**. The number of such passwords is 26^4.

(j) T F $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity element for the binary operation $\odot$ on 3×3 bit matrices.
Solution: **T**. $\mathbf{I} \odot \mathbf{A} = \mathbf{A} \odot \mathbf{I} = \mathbf{A}$ for any 3×3 bit matrix $\mathbf{A}$.