Discrete Structures, Test 2

Monday, April 6, 2009

SOLUTIONS

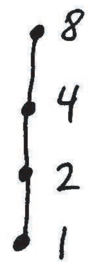
1. (12 pts) Short answer. Put your answer in the box. No partial credit.

   (a) If x, y are Boolean variables, simplify the Boolean expression $(x \vee y') \wedge (x \vee y)$. Your answer must be as simple as possible.

   Solution: $(x \vee y') \wedge (x \vee y) = x \vee (y' \wedge y) = x \vee 0 = x$.

   (b) Draw the Hasse diagram of the lattice $(D_8, |)$, where $|$ is the divisibility relation.

   Solution:



   (c) Compute $\lceil -3.52 \rceil$.

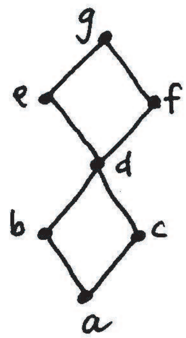
   Solution: -3.

   (d) In the lattice $(D_{30}, |)$, what is the complement of 15?

   Solution: The complement of 15 is $30/15 = 2$.

   (e) List all the maximal elements of the poset whose Hasse diagram is below.

   Solution: g.



   (f) For the poset whose Hasse diagram is above, what is the greatest lower bound of the set $B = \{c, d, f\}$?

   Solution: c.

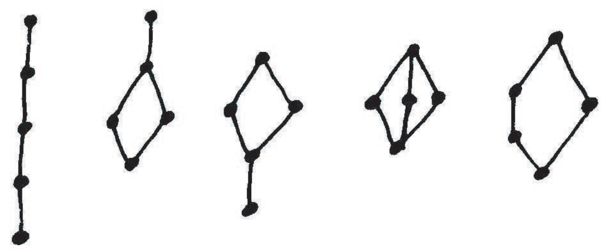
2. (8 pts) Consider the following functions
   $f_1(n) = n^3$, $f_2(n) = n \lg n$, $f_3(n) = \lg(n^{10})$, $f_4(n) = 3^n$, $f_5(n) = n!$,
   $f_6(n) = 600n^2 - 3n + 12$, $f_7(n) = n^n$, $f_8(n) = \sqrt[3]{n}$.
   Put them in order from lowest Θ class to highest (in other words, in order from slowest order of growth to fastest).

   Solution: $\Theta(\lg(n^{10})) < \Theta(\sqrt[3]{n}) < \Theta(n \lg n) < \Theta(600n^2 - 3n + 12) < \Theta(n^3) < \Theta(3^n) < \Theta(n!) < \Theta(n^n)$.

This is because $\lg(n^{10}) = 10 \lg n = \Theta(\lg n)$ is less than the Θ class of any positive power of n (such as $\sqrt[3]{n} = n^{\frac{1}{3}}$). Then $\Theta(n \lg n) > \Theta(n) > \Theta(n^{\frac{1}{3}})$ since $\Theta(n) > \Theta(1)$. Next, $\Theta(600n^2 - 3n + 12) = \Theta(n^2) > \Theta(n \lg n)$ since $\Theta(n) > \Theta(\lg n) \Rightarrow \Theta(n \cdot n) > \Theta(n \cdot \lg n)$. Then n^3 is next. After that, the exponential 3^n grows faster than any power of n. We saw in class that $\Theta(n^n) > \Theta(n!) > \Theta(b^n)$ for any base b.

3. (6 pts) Give the Hasse diagrams of all the nonisomorphic lattices that have five elements (there are 5 of them).

Solution:



Why are these the only nonisomorphic lattices with five elements? The first one is a linear order. If a lattice is not a linear order, then its Hasse diagram must contain a diamond shape (since the 2 incomparable elements must have an LUB and a GLB, to make 4 elements in a diamond shape in the Hasse diagram). Then it only remains to place the 5th element, which for a lattice can be the greatest element (the 2nd figure), the least element (the 3rd figure), incomparable to the middle two elements (the 4th figure), or can occur on a path above or below one of the incomparable elements (the last figure).

4. (6 pts) Consider the relation R on $\{1, 2, 3, 4\}$ whose matrix is $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.
Use Warshall's algorithm to compute the matrix of the transitive closure R^∞. Show your work.

Solution:

$$W_0 = \mathbf{M}_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

To compute W_1, let $k = 1$ and consider the first row and column of W_0. For the first row, we have 1's in $(k, j) = (1, 4)$ only and so $j = 4$. For the first column, we have 1's in $(i, k) = (2, 1), (4, 1)$, so $i = 2, 4$. So in W_1, we should add 1's in the $(i, j) = (2, 4), (4, 4)$

slots:

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Now let $k = 2$ and consider the second row and column of W_1. For the second row, we have 1's in $(k, j) = (2, 1), (2, 2), (2, 4)$, and so $j = 1, 2, 4$. In the second column, we have a 1 only in the $(i, k) = (2, 2)$ slot: so $i = 2$. So our new 1's are for $(i, j) = (2, 1), (2, 2), (2, 4)$, which all already have 1's in them. So

$$W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

For $k = 3$, look at the third row and the third column. For the third row we have $(k, j) = (3, 4)$ only and so $j = 4$. But there are no 1's in the third column, so there is no possible value of i, and no new 1's in W_3:

$$W_3 = W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Finally, consider $k = 4$. In the fourth row of W_3, there are 1's in $(k, j) = (4, 1), (4, 4)$, and so $j = 1, 4$. The fourth column has all 1's: so $i = 1, 2, 3, 4$. Therefore we add 1's for the values of $(i, j) = (1, 1), (2, 1), (3, 1), (4, 1), (1, 4), (2, 4), (3, 4), (4, 4)$. Therefore,

$$\mathbf{M}_{R^\infty} = W_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

5. (9 pts) Consider the relation R on $A = \{1, 2, 3, 4, 5\}$ whose matrix is

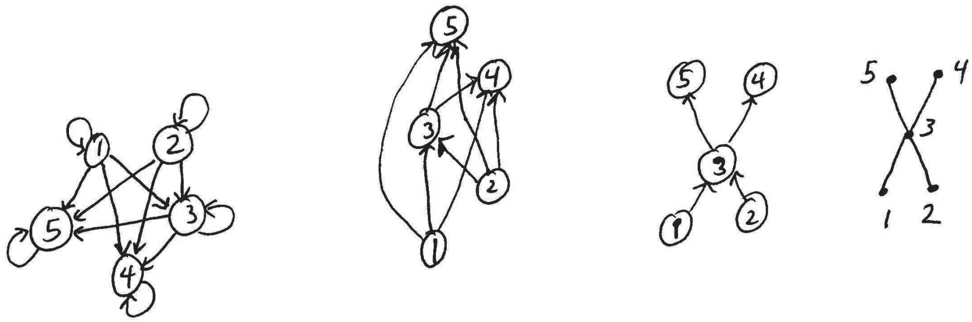
$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Show R is a partial order on A.

Solution: We need to check R is reflexive, antisymmetric, and transitive. To check R is reflexive, just note that there are all 1's on the main diagonal. To check R is antisymmetric, note that there are no matching pairs of 1's across the main diagonal. To check transitivity, look at the digraph below. We need to check that $2R3 \wedge 3R5 \Rightarrow 2R5$, $1R3 \wedge 3R5 \Rightarrow 1R5$, $1R3 \wedge 3R4 \Rightarrow 1R4$, and $2R3 \wedge 3R4 \Rightarrow 2R4$.

(b) Draw the Hasse diagram of (A, R).

Solution:

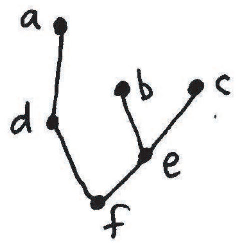


First draw the digraph, as on the left. Then remove the arrows corresponding to the reflexive loops, and make the remaining arrows all point up. Now erase the arrows that are made redundant by transitivity to get the third picture. Finally, turn this picture into a Hasse diagram.

(c) Is the poset (A, R) a lattice? Why or why not?

Solution: No. For example $\text{LUB}(4, 5)$ does not exist.

6. (20 pts) True/False. Circle T or F. No explanation needed.
For questions (a) and b below, consider the following Hasse diagram of $(A, \leq)$:



(a) T F $a \geq f$.
Solution: T.

(b) T F $(A, \leq)$ is a lattice.
Solution: F. For example, LUB(b, c) does not exist.

(c) T F For Z the set of integers, there is a function from Z to Z which is onto but not one-to-one.
Solution: T. For example $f(n) = \lfloor \frac{n}{2} \rfloor$ is such a function.

(d) T F Every function from B_n to B_1 can be represented as a Boolean polynomial.
Solution: T.

(e) T F The lattice $(D_{49}, |)$ is a Boolean algebra.
Solution: F. The prime decomposition $49 = 7^2$ contains a square.

(f) T F Warshall's algorithm, when performed on an $n \times n$ bit matrix, needs $\Theta(n^4)$ steps to be executed.
Solution: F. Warshall's algorithm takes $\Theta(n^3)$ steps.

(g) T F If $p, q \in Z^+$ are distinct prime numbers, then p and q are comparable with respect to $|$, the divisibility relation.
Solution: F. Distinct prime numbers are not comparable, since neither p nor q is divisible by the other.

(h) T F If x, y are Boolean variables, then $(x \vee y)' = x' \wedge y'$.
Solution: T. This is DeMorgan's Law.

(i) T F If A, B are sets with $|A| = 4$ and $|B| = 6$, then there is a function from A to B which is onto.
Solution: F. If A only has four elements $\{a, b, c, d\}$, then the range of any function $f(A)$ has at most four elements, since $f(A) = \{f(a), f(b), f(c), f(c)\}$.

(j) T F If A, B are sets with $|A| = 4$ and $|B| = 6$, then there is a function from A to B which is one-to-one.
Solution: T. This can be easily arranged, since for example if $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5, 6\}$, then we can choose a function g satisfying $g(a) = 1$, $g(b) = 2$, $g(c) = 3$, $g(d) = 4$, which is clearly one-to-one.