Show your work:

1. Assume the following data represents values of a function $f$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.5</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

(a) Compute the Lagrange interpolating polynomial.

**Solution:** Compute for $x_0 = 0$, $x_1 = 2$, $x_2 = 3$

$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 2)(x - 3)}{(0 - 2)(0 - 3)} = \frac{1}{6}(x^2 - 5x + 6)$,

$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{x(x - 3)}{(2 - 0)(2 - 3)} = -\frac{1}{2}(x^2 - 3x)$,

$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x - 2)}{(3 - 0)(3 - 2)} = \frac{1}{3}(x^2 - 2x)$,

$P(x) = y_0L_0(x) + y_1L_1(x) + y_2L_2(x)$

$= 2.5[\frac{1}{6}(x^2 - 5x + 6)] + 5[-\frac{1}{2}(x^2 - 3x)] + 7[\frac{1}{3}(x^2 - 2x)]$

$= \frac{5}{12}(x^2 - 5x + 6) - \frac{30}{12}(x^2 - 3x) + \frac{28}{12}(x^2 - 2x)$

$= \frac{1}{12}(3x^2 + 9x + 30) = 0.25x^2 + 0.75x + 2.5$

(b) Use part (a) to estimate $f(1)$.

**Solution:** $f(1) \sim P(1) = 0.25 \cdot 1^2 + 0.75 \cdot 1 + 2.5 = 3.5$.

(c) Use part (a) to estimate $\int_0^3 f(x) \, dx$.

**Solution:**

$$\int_0^3 f(x) \, dx \sim \int_0^3 P(x) \, dx$$

$$= (0.25 \cdot \frac{1}{3}x^3 + 0.75 \cdot \frac{1}{2}x^2 + 2.5x)^3_0$$

$$= 2.25 + 3.375 + 7.5 = 13.125$$

2. Consider the initial value problem $y' = ty + y^2 - 2$, $y(0) = 1$. Call the solution $y(t)$. 
(a) Use Euler’s method with \( h = 0.5 \) to estimate \( y(1) \).

**Solution:** \( t_0 = 0, t_1 = t_0 + h = 0.5, t_2 = t_1 + h = 1, w_0 = 1, f(t, y) = ty + y^2 - 2 \). Compute

\[
\begin{align*}
  w_1 &= w_0 + hf(t_0, w_0) = 1 + 0.5(0 \cdot 1 + 1^2 - 2) = 0.5, \\
  w_2 &= w_1 + hf(t_1, w_1) = 0.5 + 0.5(0.5 \cdot 0.5 + 0.5^2 - 2) = -0.25
\end{align*}
\]

\( w_2 \) approximates \( y(t_2) = y(1) \).

(b) Compute \( y'' \) in terms of \( t \) and \( y \).

**Solution:** Compute

\[
\begin{align*}
  y'' &= \frac{d}{dt}(ty + y^2 - 2) = 1 \cdot y + t \cdot \frac{dy}{dt} + 2y \cdot \frac{dy}{dt} - 0 \\
  &= y + (t + 2y)y' \\
  &= y + (t + 2y)(ty + y^2 - 2) \\
  &= 2y^3 + 3ty^2 + t^2y - 3y - 2t
\end{align*}
\]

Recall the book calls this \( f'(t, y) = 2y^3 + 3ty^2 + t^2y - 3y - 2t \).

(c) Use the Taylor \( n = 2 \) method with \( h = 0.5 \) to estimate \( y(1) \).

**Solution:** Compute

\[
\begin{align*}
  w_1 &= w_0 + hf(t_0, w_0) + \frac{1}{2}h^2 f'(t_0, w_0) \\
  &= 1 + 0.5(0 \cdot 1 + 1^2 - 2) \\
  &\quad + 0.5(0.5)^2(2 \cdot 1^3 + 3 \cdot 0 \cdot 1^2 + 0^2 \cdot 1 - 3 \cdot 1 - 2 \cdot 0) \\
  &= 0.375, \\
  w_2 &= w_1 + hf(t_1, w_1) + \frac{1}{2}h^2 f'(t_1, w_1) \\
  &= 0.375 + 0.5(0.5 \cdot 0.375 + 0.375^2 - 2) \\
  &\quad + 0.5 \cdot 0.5^2(2 \cdot 0.375^3 + 3 \cdot 0.5 \cdot 0.375^2 + 0.5^2 \cdot 0.375 \\
  &\quad - 3 \cdot 0.375 - 2 \cdot 0.5) \\
  &\sim -0.692871094
\end{align*}
\]

3. Consider the integral \( \int_0^2 g(x) \, dx \), where the values of \( g \) are given by the table

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2.5</td>
</tr>
</tbody>
</table>

(a) Approximate \( \int_{0}^{2} g(x) \, dx \) with \( n = 4 \) subintervals using the Midpoint Rule, the Trapezoid Rule, and Simpson’s Rule.

**Solution:** For the midpoint rule with \( n = 2 \), we have \( h = (b - a)/n = (2 - 0)/2 = 1 \) and the Midpoint Rule gives

\[
h(g(0.5) + g(1.5)) = 1(2 + 3) = 5
\]

For the trapezoid rule with \( n = 4 \), we have \( h = (b - a)/n = (2 - 0)/4 = 0.5 \), and the Trapezoid Rule gives

\[
h[0.5g(0) + g(0.5) + g(1) + g(1.5) + 0.5g(2)]
\]

\[
= 0.5(0.5(0) + 2 + 3 + 3 + (0.5)(2.5))
\]

\[
= 4.625
\]

For Simpson’s Rule with \( n = 4 \), again \( h = 0.5 \) and we have

\[
\frac{1}{3}h[g(0) + 4g(0.5) + 2g(1) + 4g(1.5) + g(2)]
\]

\[
= \frac{1}{3} \cdot 0.5(0 + 4(2) + 2(3) + 4(3) + 2.5)
\]

\[
= 4.75
\]

(b) Assume that for all \( x \in [0, 2] \), \( |g''(x)| \leq 10 \) and \( |g^{(4)}(x)| \leq 1000 \). Find the bound on the error for each of the approximations in part (a). Which is the best approximation in this case?

**Solution:** The error for the Midpoint Rule is

\[
\left| \frac{b - a}{24} h^2 |g''(\xi)| \right|
\]

for \( \xi \in [a, b] \). So this is bounded by

\[
\frac{2 - 0}{24} \cdot 1^2 \cdot 10 \sim 0.833
\]

The error for the Trapezoid Rule is

\[
\left| \frac{b - a}{12} h^2 |g''(\xi)| \right|
\]

which is bounded by

\[
\frac{2 - 0}{12} \cdot 0.5^2 \cdot 10 \sim 0.417
\]
The error for the Simpson’s Rule is
\[
\frac{b - a}{180} h^4 |g^{(4)}(\xi)|,
\]
which is bounded by
\[
\frac{2 - 0}{180} \cdot 0.5^4 \cdot 1000 \sim 0.694
\]
So in this case the Trapezoid rule may be a better approximation than Simpson’s Rule (because of the poor bound on \(g^{(4)}\)).

4. Assume that \(f(x)\) is given by the table

| \(x\) | \(0\) | 0.5 | 1 | 1.5 | 2 |
| \(f(x)\) | 6 | 10 | 12 | 9 | 4 |

(a) For each of these \(x\) values, approximate \(f'(x)\) using the most appropriate three-point rule.

**Solution:** Use the endpoint formula for \(x = 0\) (with \(h = 0.5\)) and for \(x = 2\) (with \(h = -0.5\)). For the other \(x\) values, use the 3-point midpoint formula with \(h = 0.5\).

\[
f'(0) \sim \frac{1}{2h} [ -3f(0) + 4f(0.5) - f(1) ]
\]
\[
= \frac{1}{2(0.5)} [ -3 \cdot 6 + 4 \cdot 10 - 10 ]
\]
\[
= 12
\]

\[
f'(0.5) \sim \frac{1}{2h} [ f(1) - f(0) ]
\]
\[
= \frac{1}{2(0.5)} [ 12 - 6 ]
\]
\[
= 6
\]

\[
f'(1) \sim \frac{1}{2h} [ f(1.5) - f(0.5) ]
\]
\[
= \frac{1}{2(0.5)} [ 9 - 10 ]
\]
\[
= -1
\]

\[
f'(1.5) \sim \frac{1}{2h} [ f(2) - f(1) ]
\]
\[
= \frac{1}{2(0.5)} [ 4 - 12 ]
\]
\[
= -8
\]

\[
f'(2) \sim \frac{1}{2h} [ -3f(2) + 4f(1.5) - f(1) ]
\]
\[
= \frac{1}{2(-0.5)} [ -3 \cdot 4 + 4 \cdot 9 - 12 ]
\]
\[
= -12
\]
(b) Use the three-point rule for second derivatives to approximate \( f''(1) \).

**Solution:** Choose \( h = 0.5 \), and approximate

\[
f''(1) \approx \frac{1}{h^2} [f(0.5) - 2f(1) + f(1.5)]
= \frac{1}{0.5^2} [10 - 2 \cdot 12 + 9]
= -20
\]

5. Consider the heat equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\]

with the boundary conditions \( u(t, 0) = 0 \) and \( u(t, 4) = 3 \). The initial values of \( u \) are given by

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(0, x) )</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

Use the forward difference method with \( k = 0.2 \) to approximate the \( u \) at time \( t = 0.4 \). In other words, approximate \( u(0.4, x) \) for each \( x = 0, 1, 2, 3, 4 \).

**Solution:** Compute \( \lambda = \alpha^2 \cdot k/h^2 = 1 \cdot 0.2/1^2 = 0.2 \). Note this is less than 0.5, and so the method is stable. So compute using \( w_{i,j+1} = w_{ij} + \lambda (w_{i-1,j} - 2w_{ij} + w_{i+1,j}) \), and noting that the boundary values for \( x = 0, 4 \) remain constant:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(0, x) )</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>( u(0.2, x) )</td>
<td>0</td>
<td>2.4</td>
<td>5</td>
<td>4.8</td>
<td>3</td>
</tr>
<tr>
<td>( u(0.4, x) )</td>
<td>0</td>
<td>2.44</td>
<td>4.44</td>
<td>4.48</td>
<td>3</td>
</tr>
</tbody>
</table>