1. Use the Taylor polynomials of \( \sin x \) to approximate

\[
\int_0^{\frac{1}{2}} \sin(x^2) \, dx
\]

with an error of at most \( 10^{-6} \).

(a) Let \( f(x) = \sin x \). Compute the \( n \)th derivative \( f^{(n)}(x) \). Compute

the Taylor polynomial \( P_n(x) \) of \( f(x) \) for \( x_0 = 0 \).

**Solution:** Compute \( f(x) = \sin x \), \( f'(x) = \cos x \), \( f''(x) = -\sin x \),

\( f'''(x) = -\cos x \), \( f^{(4)}(x) = \sin x \), and then the pattern repeats.

So the derivative \( f^{(n)}(x) \) depends only on the value of \( n \% 4 \) (the

remainder of \( n \) divided by 4 in Java notation). So we have

<table>
<thead>
<tr>
<th>( n % 4 )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sin x )</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \cos x )</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( -\sin x )</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( -\cos x )</td>
<td>-1</td>
</tr>
</tbody>
</table>

Assuming \( n \) is odd, the \( n \)th Taylor polynomial \( P_n(x) \) is given by

\[
P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n
\]

\[
= 0 + x + 0 + (-1)^\frac{3}{3!} + \cdots
\]

\[
= x - \frac{x^3}{3!} + \cdots \pm \frac{x^n}{n!}
\]

(If \( n \) is even, the last term is \( \pm \frac{x^{n-1}}{(n-1)!} \).)

(b) Show that the remainder term \( R_n(x) = \sin x - P_n(x) \) satisfies

\[
|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.
\]

**Solution:** The remainder term is given by

\[
R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}x^{n+1},
\]
where $\xi(x)$ is between 0 and $x$. But by the formulas above for the derivatives of $\sin x$, we know $|f^{(n+1)}(\xi(x))| \leq 1$ always. This clearly implies

$$|R_n(x)| = \frac{|f^{(n+1)}(\xi(x))|}{(n+1)!} |x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}. $$

(c) We will approximate

$$\int_0^{1/2} \sin(x^2) \, dx$$

by

$$\int_0^{1/2} P_n(x^2) \, dx.$$  
In this case, the error is given by

$$E = E_n = \left| \int_0^{1/2} R_n(x^2) \, dx \right|.$$  

(d) Use the formulas above to find an $n$ so that $E_n \leq 10^{-6}$.

**Solution:** Compute by part (b)

$$E_n = \left| \int_0^{1/2} R_n(x^2) \, dx \right|$$

$$\leq \int_0^{1/2} |R_n(x^2)| \, dx$$

$$\leq \int_0^{1/2} \frac{|x^2|^{n+1}}{(n+1)!} \, dx$$

$$= \frac{1}{(n+1)!} \cdot \frac{1}{2n+3} \cdot \left( \frac{1}{2} \right)^{2n+3}.$$  

It is too complicated to solve for $n$ if we set this quantity equal to $10^{-6}$. Instead, we just compute for the first few values of $n$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\frac{1}{(n+1)!} \cdot \frac{1}{2n+3} \cdot \left( \frac{1}{2} \right)^{2n+3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.003125</td>
</tr>
<tr>
<td>2</td>
<td>0.00186102</td>
</tr>
<tr>
<td>3</td>
<td>0.00009042</td>
</tr>
<tr>
<td>4</td>
<td>0.00000370</td>
</tr>
</tbody>
</table>

So we see that for $n = 4$, $E_4 \leq 0.000000370 < 10^{-6}$. 

(e) Compute the approximation for the value of $n$ determined in part (d).

Solution: For $n = 4$, we have

$$P_4(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}.$$ 

So our approximation is

$$\int_0^{\frac{1}{2}} P_4(x^2) \, dx = \int_0^{\frac{1}{2}} \left( x^2 - \frac{x^6}{6} \right) \, dx = \left[ \frac{x^3}{3} - \frac{x^7}{42} \right]_0^{\frac{1}{2}} \sim 0.041480655$$