(3) Let \( \{a_n\} \) be a sequence of extended real numbers. \( L \in [-\infty, \infty] \) is said to be a limit point of the sequence if there is a subsequence \( a_{n_k} \) so that \( \lim_{k \to \infty} a_{n_k} = L \). Show that \( \inf \sup_{n \geq k} a_n \) is equal to the supremum of the set of limit points of \( \{a_n\} \). This quantity is defined as \( \limsup a_n \). Show that \( \limsup a_n \) is a limit point of \( \{a_n\} \).

Hint: Here is one way to produce a convergent subsequence: If \( b_j \) is a sequence with values in \([a, \infty]\), then there is a convergent subsequence with limit in \([a, \infty]\). This is because \([a, \infty]\) is sequentially compact (as it is homeomorphic to the compact metric space \([0, 1]\)).

(4) The Heaviside function \( H(x) \) is defined to be the characteristic function \( \chi_{[0, \infty)} \).

(a) Show that there is no continuous function \( f \) that is equal to \( H \) almost everywhere.

Hint: If there were such an \( f \), consider an open neighborhood of \( f(0) \).

(b) Show that there is no sequence \( f_k \) of functions in \( C^0(\mathbb{R}) \) so that \( f_k \to H \) with respect to the \( L^\infty(\mathbb{R}) \) norm.

Hint: Prove the following fact: A complete subset of a metric space is closed.

(5) A function \( f : \mathbb{R}^d \to \mathbb{R} \) is said to vanish at infinity if \( \lim_{|x| \to \infty} f(x) = 0 \). Show that the closure in \( L^\infty(\mathbb{R}^d) \) of the set of continuous \( \mathbb{R} \)-valued functions with compact support on \( \mathbb{R}^d \) is equal to the set of continuous functions which vanish at infinity on \( \mathbb{R}^d \).

Hint: Construct a sequence \( \phi_n \) of continuous functions with compact support so that \( \phi_n \not\to 1 \) everywhere.