2 Ordinary Differential Equations

2.1 Introduction

An *ordinary differential equation* (an *ODE*) is an equation of the form

$$x^{(n)}(t) = F(x^{(n-1)}, \dots, \dot{x}, x, t), \tag{10}$$

where $x\colon I \to \mathbb{R}$ is a function of t, I is an open interval in $\mathbb{R}$,

$$\dot{x} = \frac{dx}{dt}, \quad \text{and} \quad x^{(n)} = \frac{d^n x}{dt^n}.$$

The *order* of the above equation is n, the highest derivative of x which appears. It is also useful to consider the case

$$x = (x^1, \dots, x^m)\colon I \to \mathbb{R}^m,$$

which is called a *system of ODEs.*

Some ODEs can be solved explicitly by using integration techniques, but most cannot. For most ODEs, instead of explicit solutions, we must rely on an abstract *existence* theorem to show that for nice enough F (Lipschitz suffices), there is a *unique* solution locally. We also investigate the *regularity* of solutions, showing, for example, if F is smooth, then any solution to (10) is smooth. Existence, uniqueness, and regularity are three main themes in the theory of all differential equations, and there are satisfactory theorems to handle all three for ODEs.

Consider the following example (where x, not t, is the dependent variable):

Example 5. *Consider the differential equation $dy/dx = x^2 y$. This first order ODE is called* separable*, since it is written in the form $dy/dx = f(x)g(y)$. Recall the solution procedure for a separable ODE:*

- *If c is a root of $g(y)$, then $y = c$ is a solution. (Why?) So in the present case, $y = 0$ is a solution.*

- *For other values of $g(y)$, compute*

$$\begin{aligned}
\frac{dy}{dx} &= x^2 y, \\
\frac{dy}{y} &= x^2\,dx, \\
\int \frac{dy}{y} &= \int x^2\,dx, \\
\ln|y| &= \frac{x^3}{3} + C, \\
y &= \pm e^C e^{x^3/3} = C' e^{x^3/3},
\end{aligned}$$

  where $C' = \pm e^C$ is a nonzero constant.

- *If we let C' be any real number, then we capture both cases above, and the general solution is $y = C' e^{x^3/3}$.*

Homework Problem 14. *Consider the ODE*

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}.$$

(a) Find the general solution to this differential equation. Your answer should be rational functions of x. You may need to write your answer using more than one case.

(b) Find the particular solution passing through $(x, y) = (1, 1)$.

(c) Find the particular solution passing through $(x, y) = (1, -1)$. (Hint: What is the formula for $\tan(\phi + \frac{\pi}{2})$?)

2.2 Local Existence and Uniqueness

The most natural setting for systems of ODEs is in terms of an *initial value problem.* Let $x = (x^1, \ldots, x^n) = x(t)$. An initial value problem for a first order system of ODEs at $t = t_0$ consists of

- a system of ODEs $\dot{x} = v(x, t)$
- and an initial condition $x(t_0) = x_0$.

We'll see below that if v satisfies a Lipschitz condition, and for t in a small interval around t_0, there is a unique solution to the initial value problem.

Example 6. *Consider the following problem: Find a solution to the ODE* $\dot{y} = y^2$ *subject to the initial condition* $y(0) = 1$. *Interpreting* t *as a time variable, what happens as time goes forward from* $t = 0$*?*

Solution: $dy/dt = y^2$ *is separable, and so compute*

$$\frac{dy}{y^2} = dt \Longrightarrow \int \frac{dy}{y^2} = \int dt \Longrightarrow -\frac{1}{y} = t + C \Longrightarrow y = -\frac{1}{t+C}.$$

Plug in the initial condition $y = 1$ *and* $t = 0$ *to solve for* C *to find* $C = -1$ *and*

$$y = \frac{1}{1-t}.$$

Note that $y(t)$ *is discontinuous at* $t = 1$, *so as time goes forward from* $t = 0$, *the solution only exists until time 1. Also note there is no problem going backward in time, and so the solution to the initial value problem is*

$$y = \frac{1}{1-t}, \qquad t \in (-\infty, 1).$$

It does not make sense to talk about the solution to the initial value problem beyond $t = 1$.

The previous example shows that it is not in general possible to extend a solution to an initial value problem for all time. However, we can still hope to find a solution to an initial value problem on a neighborhood $(t_0 - \epsilon, t_0 + \epsilon)$ of t_0.

Theorem 5. *Consider the initial value problem*

$$\begin{cases} \dot{x} = v(x,t), \\ x(t_0) = x_0 \end{cases} \tag{11}$$

for $x : I \longrightarrow \mathbb{R}^n$ *for* I *an open neighborhood of* t_0. *Assume* v *is a Lipschitz function from* $\mathcal{O} \times I \rightarrow \mathbb{R}^n$, *where* $\mathcal{O} \subset \mathbb{R}^n$ *is an open neighborhood of* x_0. *Then on a neighborhood* $\tilde{I}$ *of* t_0 *contained in* I, *there is a unique solution* ϕ *to (11).*

Before we give the proof, let us consider a few examples.

Example 7. *The differential equation* $\dot{x} = x^2 + t$ *has no solution which can be written down in terms of standard algebraic and transcendental functions (such as roots, exponentials, trigonometric functions). Theorem 5 states that there is a local solution for every initial value problem. For example, for initial conditions* $x(0) = 1$*, there is a solution valid on an open interval containing* $t = 0$*.*

Theorem 5 does not guarantee a solution which is valid for all time t *(see Example 6 above). In fact the solution for the present initial-value problem will also blow up in finite time. This is basically because for* $t \geq 0$, $\dot{x} = x^2 + t \geq x^2$*, and so the solution should grow faster than the solution to Example 6, which goes to infinity in finite time.*

If v in Theorem 5 is not Lipschitz, then it is possible to lose the uniqueness statement from Theorem 5 (although existence is still valid).

Example 8. *Consider the initial value problem*

$$\dot{x} = x^{\frac{2}{3}}, \qquad x(0) = 0.$$

Then it is straightforward to verify that $x(t) = 0$ *is a solution. There is another solution, however. Solve the equation*

$$\begin{aligned} \frac{dx}{dt} &= x^{\frac{2}{3}}, \\ x^{-\frac{2}{3}}dx &= dt, \\ \int x^{-\frac{2}{3}}dx &= \int dt, \\ 3x^{\frac{1}{3}} &= t + C, \\ x &= (\tfrac{1}{3}t + \tfrac{1}{3}C)^3. \end{aligned}$$

Then plug in $x(0) = 0$ *to find* $C = 0$ *and the solution* $x(t) = (\frac{1}{3}t)^3$*.*

The point of this example is that $v = x^{\frac{2}{3}}$ *is not Lipschitz—see Example 4 above. Therefore, Theorem 5 does not apply.*

Proof of Theorem 5. The idea of the proof is to set up the problem in terms of a contraction mapping. We first find an iteration whose fixed point solves the differential equation and then find an appropriate complete metric space on which the iteration is a contraction map.

For a continuous $\mathbb{R}^n$-valued function ϕ defined on a neighborhood of t_0, let $A\phi$ be another such function defined as follows:

$$(A\phi)(t) = x_0 + \int_{t_0}^{t} v(\phi(\tau), \tau) d\tau. \tag{12}$$

(Note we are integrating $\mathbb{R}^n$-valued function. This may be related t o the usual $\mathbb{R}$-valued integration theory by considering each component separately.) A will be our iterative map, and we consider ϕ, $A\phi$, $A^2\phi$, etc., to be the *Picard approximations* for the initial value problem. We consider Picard approximations because of the following

Lemma 18. *A continuous fixed point of the Picard approximation (12) is a solution to the initial value problem (11). In particular, any such fixed point is continuously differentiable.*

Proof. If $A\phi = \phi$, then compute

$$\dot{\phi} = \frac{d}{dt}\left[x_0 + \int_{t_0}^{t} v(\phi(\tau), \tau) d\tau\right] = v(\phi(t), t)$$

by the Fundamental Theorem of Calculus. In particular, since ϕ and v are continuous (Lemma 13), $\dot{\phi}$ is continuous, and so ϕ is continuously differentiable. Lastly, check the initial condition

$$\phi(t_0) = x_0 + \int_{t_0}^{t_0} v(\phi(\tau), \tau) d\tau = x_0$$

to complete the proof of the lemma. □

Our complete metric space will be

$$X = \{\phi \in C^0(\tilde{I}, \mathbb{R}^n) : \phi(t_0) = x_0, \sup_{t \in \tilde{I}} |\phi(t) - x_0| \le P\},$$

where $\tilde{I} = [t_0 - \epsilon, t_0 + \epsilon] \subset I$ for a small positive ϵ to be determined later, $|\cdot|$ is the norm on $\mathbb{R}^n$, and P is chosen so that the closed ball $\overline{B_{x_0}(P)} = \{x : |x - x_0| \le P\} \subset \mathcal{O}$. We first demonstrate

Lemma 19. *X is a complete metric space.*

Proof. First of all, $C^0(\tilde{I}, \mathbb{R}^n)$ is complete by Proposition 1. Moreover, the conditions imposed give closed subsets of the Banach space C^0. The second condition is obviously closed since the norm on any Banach space is continuous. To check the condition $\phi(t_0) = x_0$ is closed, use the following lemma, whose proof is immediate:

Lemma 20. *For a metric space J and $y \in J$, the map from the Banach space $C^0(J, \mathbb{R}^n)$ to $\mathbb{R}^n$ given by $f \mapsto f(y)$ is continuous.*

Since these two conditions are closed, X is a closed subset of the complete metric space $C^0(\tilde{I}, \mathbb{R}^n)$, and so is complete with the induced metric. ☐

Remark. Lemma 20 is false for the Banach space L^∞. Why?

So we have proved that X is a complete metric space. Next we show

Lemma 21. *For $\epsilon > 0$ small enough, $A \colon X \to X$.*

Proof. First of all, choose $\delta > 0$ so that $[t_0 - \delta, t_0 + \delta] \subset I$. Since v is continuous and $\{x : |x - x_0| \leq P\} \times [t_0 - \delta, t_0 + \delta]$ is compact, there is a constant M so that

$$\sup_{|t-t_0|\leq\delta,\, |x-x_0|\leq P} |v(x,t)| \leq M.$$

In order for this bound to work below, we must have $\epsilon \leq \delta$ (so then $\tilde{I} \subset [t_0 - \delta, t_0 + \delta]$). To check $A \colon X \to X$, we need to check for each $\phi \in X$,

1. $A\phi$ is continuous. This follows as in Lemma 18 above.
2. $(A\phi)(t_0) = x_0$. This is easy to check as in Lemma 18.
3. $\sup_{t\in\tilde{I}} |(A\phi)(t) - x_0| \leq P$. To check this, write

   $$|(A\phi)(t) - x_0| = \left| \int_{t_0}^{t} v(\phi(\tau), \tau)\, d\tau \right| \leq M|t - t_0| \leq M\epsilon,$$

   where we have used the fact that $\phi \in X$ and the definition of M to show the first inequality. So this condition is satisfied if $\epsilon \leq P/M$.

So $A \colon X \to X$ if $\epsilon \leq \min\{\delta, P/M\}$. ☐

Finally we use the Lipschitz hypothesis on v to show that A is a contraction map. Let L be the Lipschitz constant for v. Then for $\phi, \psi \in X$, compute

$$\begin{aligned}
|(A\phi)(t) - (A\psi)(t)| &= \left| \int_{t_0}^{t} [v(\phi(\tau), \tau) - v(\psi(\tau), \tau)] d\tau \right| \\
&\leq \int_{t_0}^{t} |v(\phi(\tau), \tau) - v(\psi(\tau), \tau)| d\tau \\
&\leq \int_{t_0}^{t} L|\phi(\tau) - \psi(\tau)| d\tau \\
&\leq L\|\phi - \psi\|_{C^0}|t - t_0| \\
&\leq \epsilon L\|\phi - \psi\|_{C^0}
\end{aligned}$$

Then since $\|A\phi - A\psi\|_{C^0} = \sup_{t\in\tilde{I}} |(A\phi)(t) - (A\psi)(t)|$, we see that

$$\|A\phi - A\psi\|_{C^0} \leq \epsilon L\|\phi - \psi\|_{C^0}.$$

So A is a contraction map if $\epsilon < 1/L$. Thus all together, if we require $\epsilon < \min\{\delta, P/M, 1/L\}$, then A is a contraction map on X, and its fixed point is a solution to the initial value problem.

In order to show uniqueness of the initial value problem, note that the Contraction Mapping Theorem automatically proves that any two continuous solutions ϕ_1 and ϕ_2 to the initial value problem from $\tilde{I}$ to $\mathbb{R}^n$ must coincide if the additional constraint

$$\sup_{t\in\tilde{I}} |\phi(t) - x_0| \leq P$$

is satisfied. Since ϕ_1 and ϕ_2 are continuous and satisfy the initial condition, this condition is automatically satisfied for both ϕ_1 and ϕ_2 on a (perhaps smaller) interval $\hat{I} \subset \tilde{I}$ containing t_0. Then uniqueness applies on this smaller interval, since A is a contraction map for any ϵ small enough. Note that the interval $\hat{I}$ on which $\phi_1 = \phi_2$ may depend on ϕ_1 and ϕ_2. The proof that the two solutions must coincide on all of $\tilde{I}$ depends on the Extension Theorem 6 below. □

We record what we have proven so far with respect to uniqueness here.

Proposition 22. *Any two solutions ϕ_1 and ϕ_2 to the initial value problem (11) coincide on a small interval containing t_0. The interval may depend on the solutions ϕ_1 and ϕ_2*

Remark. Note that in the proof of the previous theorem, we only use that v is Lipschitz in the x variables (with a uniform Lipschitz constant uniform valid for all t). We still require v to be continuous in t.

The previous theorem provides a continuously differentiable solution on an interval $\tilde{I}$ containing the initial time t_0 and proves uniqueness on a (perhaps) smaller interval $\hat{I}$. There is a satisfactory more global theory of ODEs which we detail in the next subsection.

2.3 Extension of solutions

Recall, from Corollary 16 above, that any locally C^1 function f from Ω, a domain in $\mathbb{R}^n$, to $\mathbb{R}^m$ is locally Lipschitz. In other words, f is Lipschitz when restricted to any compact subset of Ω.

Theorem 6 (Extension). *Consider an initial value problem*

$$\dot{x} = v(x,t), \qquad x(t_0) = x_0. \tag{13}$$

Assume v is continuous and locally Lipschitz in $\mathbb{R}^n \times I$, where I is an open interval containing t_0. Then there is an open interval J satisfying $t_0 \in J \subset I$ and a unique solution $\phi\colon J \to \mathbb{R}^n$ to the initial value problem. Moreover, J is maximal in the following sense: if there is a time $T \in I \cap \partial J$, then $\limsup_{t\to T} |\phi(t)| = \infty$.

So this theorem says that if we start with an initial condition $x(t_0) = x_0$ and flow forward (or backward) in time by satisfying the ODE, then there is a unique solution which continues until (1) the end of the interval I is reached, or (2) the solution blows up.

Proof. We first consider the following lemma, which is a consequence of the proof of Theorem 5 above:

Lemma 23. *On any compact subset K of $\mathbb{R}^n \times I$, there is an $\epsilon > 0$ so that for any $(x_0, t_0) \in K$, there exists a solution to the initial value problem $\dot{x} = v(x,t)$, $x(t_0) = x_0$ which is valid on $[t_0 - \epsilon, t_0 + \epsilon]$.*

The point is that there is a uniform ϵ which works for all initial conditions $(x_0, t_0) \in K$.

Proof. Recall that in the proof of Theorem 5. Any $\epsilon < \min\{\delta, P/M, 1/L\}$ works. By compactness of K and since I is open, we can choose a uniform $\delta > 0$ so that for all $(x_0, t_0) \in K$, $[t_0 - \delta, t_0 + \delta] \subset I$. We may choose P to be any positive number (since $\mathcal{O} = \mathbb{R}^n$ in the present case). The Lipschitz constant $L = L_{\tilde{K}}$ is uniform over any compact set $\tilde{K}$ by the locally Lipschitz property of f. Let

$$M = \max_{(x,t)\in\tilde{K}} |v(x,t)|,$$

where

$$\tilde{K} = \{(x,t) \in \mathbb{R}^{n+1} : \exists (x_0,t_0) \in K : |t - t_0| \leq \delta, |x - x_0| \leq P\}$$

It is straightforward to check $\tilde{K}$ is compact (it is the image of the compact set $K \times \overline{B_P(0)} \times [-\delta, \delta] \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ under the continuous map $+ \colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.) Therefore, since v is continuous, M can be chosen independently of $(x_0, t_0) \in K$.

(Note the reason we need to go to all of $\tilde{K}$: the definition of M in the proof of Theorem 5 above is

$$M = \sup_{|t-t_0|\leq\delta, |x-x_0|\leq P} |v(x,t)|.$$

In order to have a single M work for all $(x_0, t_0) \in K$, we must have let $(x,t) \in \tilde{K}$. L must be valid on all of $\tilde{K}$ as well, since we consider integrals from t_0 to t, where $(x_0, t_0) \in K$, $|t - t_0| \leq \epsilon < \delta$.)

Now we must ensure that $\epsilon < \min\{\delta, P/M, 1/L\}$. All of these quantities can be chosen independently of $(x_0, t_0) \in K$. ☐

Lemma 24 (Gluing solutions). *Consider any two solutions to $\dot{x} = v(x,t)$ which are defined on intervals in $\mathbb{R}$. If the two coincide on any interval in $\mathbb{R}$ then they must coincide on the entire intersection of their intervals of definition. Thus they can be glued together to form a solution on the union of their intervals of definition.*

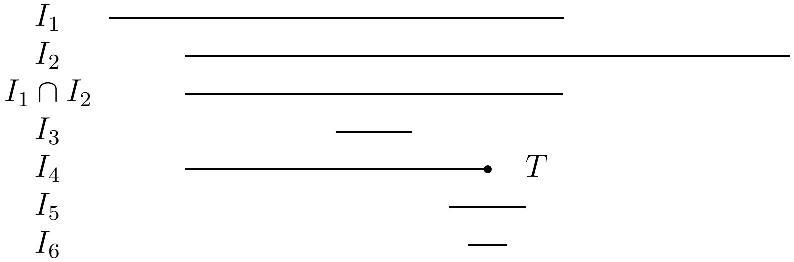
Proof. Consider two solutions ϕ_1, ϕ_2 to $\dot{x} = v(x,t)$ defined on intervals I_1 and I_2. Assume they coincide on an interval $I_3 \subset I_1 \cap I_2$. We want to show $\phi_1 = \phi_2$ on all of $I_1 \cap I_2$. Let I_4 be the largest interval containing I_3 on which ϕ_1 and ϕ_2 coincide (take I_4 to be the path-connected component of the closed set $\{t : \phi_1(t) = \phi_2(t)\}$ containing I_3). Now we will show that $I_4 = I_1 \cap I_2$.

Assume $I_4 \neq I_1 \cap I_2$. Then since I_4 is a relatively closed subinterval of $I_1 \cap I_2$, there is an endpoint T of I_4 in the interior of $I_1 \cap I_2$. Theorem 5 says there is a solution ϕ_3 to

$$\dot{x} = v(x,t), \qquad x(T) = \phi_1(T) \;\; [= \phi_2(T)]$$

on an interval I_5 centered at T. ϕ_1 and ϕ_2 also satisfy this initial value problem, and by the uniqueness of ϕ_3 (Proposition 22), we must have $\phi_1 = \phi_3 = \phi_2$ on a (perhaps smaller) interval $I_6 \ni t_0$. Thus I_4 must contain I_6 and we have a contradiction to the assumption that T is an endpoint of I_4 in the interior of $I_1 \cap I_2$. Thus $I_4 = I_1 \cap I_2$.

It may help to refer to the following picture of the intervals involved.



Now we have proved that $\phi_1 = \phi_2$ on the intersection of their domains of definition $I_1 \cap I_2$. To extend to $I_1 \cup I_2$, define

$$\phi(t) = \begin{cases} \phi_1(t) \text{ for } t \in I_1, \\ \phi_2(t) \text{ for } t \in I_2 \setminus I_1. \end{cases}$$

Note that ϕ is a solution to the differential equation since both ϕ_1 and ϕ_2 are. There is no trouble with the differentiability of this piecewise-defined function since $\phi_1 = \phi_2$ on the whole interval $I_1 \cap I_2$. □

For simplicity, consider only solutions moving forward in time. Let

$$E = \{t \in I_+ : \text{ there is a unique solution } \phi \text{ to (13) on } [t_0, t)\},$$

where $I_+ = I \cap (t_0, \infty)$. We will set this E to be equal to $J_+ = J \cap (t_0, \infty)$. Uniqueness on $[t_0, t)$ means any other solution to the initial value problem defined on an interval containing $[t_0, t)$ must coincide with ϕ there. It will suffice to prove the following

Lemma 25. *If* $\sup_E |\phi| \leq C < \infty$, *then* $E = I_+$.

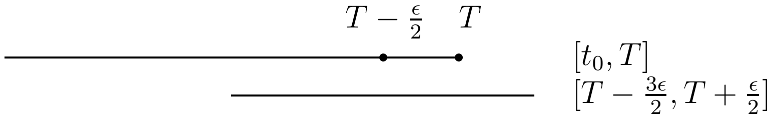
Proof. Assume $|\phi|$ is uniformly bounded on E. Then to prove the lemma it is enough to show that E is a nonempty, open, and closed subset of I_+ (and so $E = I_+$ since I_+ is connected). E is nonempty by Theorem 5 above.

To show E is open in I_+, let $T \in E$. Then there is a unique solution ϕ defined on (t_0, T). First we note that $(t_0, T] \subset E$. To see this, let $T' \in (t_0, T]$. Then the restriction of $\phi = \phi_T$ to $[t_0, T']$ is a solution to (13) on $[t_0, T)$. Moreover, it is unique, since any other solution to (13) on $[t_0, T')$ agrees with ϕ on a neighborhood of t_0, and so Lemma 24 shows they must agree on all $[t_0, T')$.

So to show E is open, we may restrict our attention to times larger than T. Since $|\phi|$ is uniformly bounded by C and $[t_0, T]$ is a compact subinterval of I, we may apply Lemma 23 to show there is uniform ϵ so that any solution to the differential equation with initial condition $x(\tau) = \chi$ for $\tau \in [t_0, T]$, $|\chi| \leq C$ must exist on $[\tau - \epsilon, \tau + \epsilon]$. Now we may consider the initial value problem

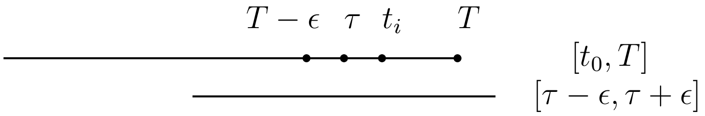
$$\dot{x} = v(x, t), \qquad x(T - \tfrac{\epsilon}{2}) = \phi(T - \tfrac{\epsilon}{2}). \tag{14}$$

So Lemma 23 shows there is a solution $\tilde{\phi}$ to this initial value problem which exists on $[T - \frac{3\epsilon}{2}, T + \frac{\epsilon}{2}]$. Moreover, Lemma 24 says that $\phi = \tilde{\phi}$ on the intersection of their intervals of definition, and moreover, that ϕ may be extended by $\tilde{\phi}$ to a solution on $[t_0, T + \frac{\epsilon}{2}]$. Lemma 24 also implies this extension is unique on every subinterval containing t_0, and so in particular $[T - \frac{\epsilon}{2}, T + \frac{\epsilon}{2}] \subset E$ and E is open.



It remains to show that E is closed in I_+. Let $T \in \bar{E} \cap I_+$. Let $t_i \in E$, $t_i \to T$. Then the assumption that $|\phi| \leq C$ on E implies there is a uniform ϵ so that for all t_i, there is a solution on $[t_i - \epsilon, t_i + \epsilon]$. Choose t_i so that $|T - t_i| < \epsilon$. Also, let $\tau < t_i$ so that $|T - \tau| < \epsilon$. Now we use the same argument as in previous paragraphs: Use the solution ϕ on $[t_0, t_i)$ to construct a solution $\tilde{\phi}$ on $[\tau - \epsilon, \tau + \epsilon] \ni T$. Lemma 24 allows us to glue ϕ and $\tilde{\phi}$ together to form a unique solution valid on $[t_0, \tau + \epsilon] \ni T$. So $T \in E$ as above and E

is closed in I_+.



□

This Lemma 25 completes the proof of the Extension Theorem 6, at least for solutions moving forward in time. The reason is this: if there is a time $T \in I_+ \cap \partial J$ (we may choose I_+ since we are only moving forward in time), then

$$E = J_+ \neq I_+.$$

Therefore, by the contrapositive of Lemma 25, $\sup_E |\phi| = \infty$. But since ϕ is continuous on $[t_0, T)$, we must have $\limsup_{t\to T} |\phi(t)| = \infty$.

The argument for solutions moving backward in time is the same. □

Remark. The above theorem may be improved as follows:

Theorem 7. Consider an initial value problem $\dot{x} = v(x,t)$, $x(t_0) = x_0$. Assume v is continuous and locally Lipschitz in $\mathcal{O} \times I$, where $\mathcal{O}$ is a connected open subset of $\mathbb{R}^n$ and I is an open interval containing t_0. Then there is an open interval J satisfying $t_0 \in J \subset I$ and a unique solution $\phi\colon J \to \mathbb{R}^n$ to the initial value problem. Moreover, J is maximal in the following sense: if there is a time $T \in I \cap \partial J$, then as $t \to T$, $\phi(t)$ leaves every compact subset of $\mathcal{O}$.

The proof is essentially the same as that of Theorem 6.

Here is an important principle which follows from the basic theorems

Proposition 26. *Consider the graph of a solution $(t, x(t))$ to a differential equation $\dot{x} = v(x,t)$, where v is Lipschitz. If any two solutions have graphs which cross, then they must coincide on the intersection of their intervals of definition.*

Proof. Let ϕ_1 and ϕ_2 be the two solutions. If their graphs cross at (t_0, x_0), then they both solve the initial value problem

$$\dot{x} = v(x,t), \qquad x(t_0) = x_0.$$

The solutions must coincide on a small interval by Theorem 5, and then must coincide on the whole intersection of their intervals of definition by Lemma 24. □

Homework Problem 15. *Consider the initial value problem* $\dot{x} = x^2 + t$, $x(0) = 1$. *Show that the solution to this problem (moving forward in time) exists only until some time* $T > 0$, *where* $T < 1$.

Hint: See Examples 6 and 7 above. Let $\phi(t)$ *be the solution to the current initial value problem. We will compare* ϕ *to the solution* $\psi(t) = \frac{1}{1-t}$ *of the initial value problem* $\dot{x} = x^2$, $x(0) = 1$. *Let* J *be the maximal interval on which* ϕ *can be extended. Let* $J_+ = J \cap (0, \infty)$; T *is then the positive endpoint of* J_+. *Now consider the interval*

$$E = \{t \in J_+ : \phi(\tau) \geq \psi(\tau) \text{ for all } \tau \in (0, t]\}.$$

- *(a) Show that* $E = J_+$ *implies* $T \leq 1$. *(Use Theorem 6.)*
- *(b) Proceed to show* $E = J_+$. *It suffices to show* E *is nonempty, open and closed in* J_+. *Why?*
- *(c) To show* E *is nonempty, differentiate the equation* $\dot{\phi} = \phi^2 + t$ *at* $t = 0$. *This will allow you to compute* $\ddot{\phi}(0)$. *Show that* $\phi(0) = \psi(0)$, $\dot{\phi}(0) = \dot{\psi}(0)$, *and* $\ddot{\phi}(0) > \ddot{\psi}(0)$. *Why does this show* E *is nonempty? (Use Taylor's Theorem or integrate in* t *twice; in particular, by the regularity results in Subsection 2.5 below,* $\ddot{\phi}$ *is continuous.)*
- *(d) To show* E *is open, show that* $\dot{\phi}(t) > \dot{\psi}(t)$ *for* $t \in E$.
- *(e) To show* E *is closed, use the continuity of* ϕ *and* ψ. *So this proves* $E = J_+$ *and so* $T \leq 1$.
- *(f) To show* $T < 1$, *note that part* (c) *implies there is a point* $\tau \in E$ *where* $\phi(\tau) > \psi(\tau)$. *Let* $\tilde{\psi}(t)$ *be the solution to the initial value problem* $\dot{x} = x^2$, $x(\tau) = \phi(\tau)$. *Solve this equation explicitly and show that* $\tilde{\psi}$ *blows up at a time* $\tilde{T} < 1$. *Then note that parts* (a)-(e) *can be repeated to show that* $J_+ \subset (0, \tilde{T})$.

2.4 Linear systems

If $x \in \mathbb{R}^n$, a *homogeneous linear system* is a system of the form $\dot{x} = A(t)x$, where $A(t)$ is an $n \times n$ matrix valued function of t alone. In this case, it is straightforward to see that the space of solutions is a vector space over $\mathbb{R}$. In other words, if $\alpha \in \mathbb{R}$, ϕ, ψ satisfy the equation, then $\alpha\phi + \psi$ also satisfies the equation. The existence and uniqueness theorem allows us to find the dimension of the solution space.

Proposition 27. *Consider the equation* $\dot{x} = A(t)x$*, where* $A(t)$ *is a continuous* $n \times n$ *matrix valued function of* t*, and* $x(t) \in \mathbb{R}^n$*. For each* t_0*, there is an interval* $I \ni t_0$ *so that the space of solutions* $\phi(t)$ *on* I *has dimension* n*. Consider an initial value condition* $x(t_0) = x_0$*. Let* $\phi_{x_0}(t)$ *be the solution to this initial value problem. Then the map* $\mathcal{S}\colon x_0 \mapsto \phi_{x_0}$ *is a linear isomorphism from* $\mathbb{R}^n$ *to the space of solutions defined on* I*.*

Remark. It is not too hard to show that the interval I can be taken to be the maximal open interval containing t_0 on which $A(t)$ is continuous. (See Michael Taylor, Partial Differential Equations, Basic Theory.)

Proof. $A(t)x$ is locally Lipschitz in x and continuous in t, as needed for Theorems 5 and 6. First of all, for a basis ξ_i of $\mathbb{R}^n$, let I be a small interval on which all the solutions ϕ_{ξ_i} exist. Note the map $x_0 \mapsto \phi_{x_0}$ is obviously linear. $\mathcal{S}$ is injective since if $x_0 \neq y_0$, $\phi_{x_0}(t_0) \neq \phi_{y_0}(t_0)$, and thus $\phi_{x_0} \neq \phi_{y_0}$. Therefore, if $x_0 = a^i \xi_i$, $\phi_{x_0} = a^i \phi_{\xi_i}$. Again by uniqueness, any solution ϕ to $\dot{x} = A(t)x$ is determined by the initial value $\phi(t_0) = x_0$, and so $\mathcal{S}$ is onto. □

Given a linear equation $\dot{x} = A(t)x$, for $x = x(t) \in \mathbb{R}^n$, we can consider a similar equation $\dot{X} = A(t)X$ for $X = X(t)$ an $n \times n$ matrix valued function. The solution $\Phi(t)$ of the initial value problem

$$\dot{X} = A(t)X, \quad X(t_0) = I \text{ the identity matrix,}$$

is called the *fundamental solution* of the equation $\dot{x} = A(t)x$. It is straightforward to see that the i^{th} column of $\Phi(t)$ is the solution to $\dot{x} = A(t)x$, $x^j(t_0) = \delta^j_i$. Moreover, the fundamental solution can be used to compute any solution to the differential equation near t_0.

Lemma 28. *On the maximal interval of existence of the fundamental solution* $\Phi(t)$ *of* $\dot{x} = A(t)x$*, the solution to the initial value problem*

$$\dot{x} = A(t)x, \quad x(t_0) = x_0,$$

is given by $\Phi(t)x_0$*.*

Proof. The proof is an immediate calculation. □

Homework Problem 16. *An* inhomogeneous linear system *is a system of the form*

$$\dot{x} = A(t)x + b(t), \tag{15}$$

where $A(t)$ *and* x *are as above and* $b(t)$ *is a continuous* $\mathbb{R}^n$*-valued function.*

(a) Let $\psi(t)$ be a solution to (15). Show that the solution space to (15) is equal to

$$\{\psi(t) + \phi(t) : \phi(t) \text{ solves } \dot{x} = A(t)x\}.$$

(b) In dimension 1, let $\Phi(t)$ be the fundamental solution to $\dot{x} = A(t)x$. Show that the general solution to (15) is

$$\Phi(t)\left[\int \frac{b(t)}{\Phi(t)}\,dt + C\right].$$

(c) Still in dimension 1, solve the initial value problem $\dot{x} = x + t$, $x(0) = 1$.

An important example class of equations are those with constant coefficients. $\dot{x} = Ax$, for A a constant $n \times n$ matrix. The fundamental solution to such an equation (with $t_0 = 0$) can be calculated directly. In the case that A is diagonalizable, write $A = PDP^{-1}$, with $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ the diagonal matrix with the eigenvalues λ_i of A along the diagonal and P the matrix whose columns are a basis of eigenvectors for the appropriate eigenvalues. Then if we define

$$e^{tD} = \operatorname{diag}(e^{t\lambda_1}, \dots, e^{t\lambda_n}),$$

then the fundamental solution to $\dot{x} = Ax$ is given by

$$e^{tA} \equiv Pe^{tD}P^{-1}.$$

To check that e^{tA} is the fundamental solution, note that $e^{0A} = I$ and

$$\begin{aligned}
\frac{d}{dt}e^{tA} &= \frac{d}{dt}(Pe^{tD}P^{-1}) \\
&= P\left(\frac{d}{dt}e^{tD}\right)P^{-1} \\
&= PDe^{tD}P^{-1} \\
&= PDP^{-1}Pe^{tD}P^{-1} \\
&= Ae^{tA}.
\end{aligned}$$

One thing to note is D and P may be complex-valued matrices. This doesn't cause any problem if we use Euler's formula

$$e^{x+iy} = e^x(\cos y + i \sin y).$$

Not every matrix B is diagonalizable. To find a general formula for the fundamental solution e^{tB}, we need to deal with the case of Jordan blocks. The following problem addresses this.

Homework Problem 17. *Let B be the $n \times n$ Jordan block matrix*

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \tag{16}$$

with λ on the diagonal, 1 just above the diagonal, and 0 elsewhere. Find the fundamental solution e^{tB} to $\dot{x} = Bx$.

Hint: Write out the system of equations in terms of components. Note that $\dot{x}^n$ only involves x^n and not any other x^i. So first solve the appropriate initial value problems for x^n (you'll need to do one initial value problem for each column of the identity matrix I). Then do x^{n-1}, then x^{n-2}, etc., and find a formula that works for all x^i.

Alternatively, it is possible to write out e^{tB} as a power series. If you approach the problem this way, you must check to be sure your answer works.

Of course the reason we consider Jordan blocks is the following famous theorem.

Theorem 8 (Jordan Canonical Form). *Let A be an $n \times n$ complex matrix. Then we can write $A = PBP^{-1}$, where B is an upper triangular, block diagonal matrix of the form*

$$B = \begin{pmatrix} B_1 & 0 & 0 & \cdots & 0 \\ 0 & B_2 & 0 & \cdots & 0 \\ 0 & 0 & B_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_m \end{pmatrix},$$

where each B_i is an $l_i \times l_i$ Jordan block matrix of the form (16) for $i = 1, \dots, m$, $\lambda = \lambda_i$ an eigenvalue of A. Of course $l_1 + \cdots + l_m = n$. If λ is a root of the characteristic polynomial $\det(\lambda I - A)$ *repeated k times, then*

$$\sum_{\lambda_i = \lambda} l_i = k.$$

B is unique up to the ordering of the blocks B_i.

Remark. A is diagonalizable if and only if each Jordan block is 1×1. If the characteristic polynomial of A has distinct roots, then A is diagonalizable, but the converse is false in general ($A = I$ the identity matrix is a counterexample).

Homework Problem 18. *Assume that all the eigenvalues of the* $n \times n$ *matrix* A *have negative real part. (*A *is not necessarily diagonalizable.) Show that* $e^{tA} \to 0$ *as* $t \to \infty$*. (Just check that each entry in the matrix* e^{tA} *goes to 0.)*

Homework Problem 19. *Solve the initial value problem*

$$\dot{x} = 2x - y, \quad \dot{y} = 2x + 5y, \quad x(0) = 2, \quad y(0) = 1.$$

2.5 Regularity

Regularity of a function refers to how many times the function may be differentiated. A function is (locally) C^k if it and all of its partial derivatives up to order k are continuous. A function is C^∞ if it and all of its partial derivatives of all orders are continuous. For the purposes of this course, a function is *smooth* if it is C^∞ (in other settings a function may be called smooth if it has as many derivatives as the purpose at hand requires). There are other notions of regularity in which the function and perhaps its derivatives, suitably defined, are in L^p or other Banach spaces.

A vector-valued function is smooth or C^k if and only if each of its component functions is smooth or C^k respectively.

Theorem 9. *Assume* $v \colon \mathcal{O} \times I \to \mathbb{R}^n$ *is smooth (*$\mathcal{O} \subset \mathbb{R}^n$ *is a domain and* $I \subset \mathbb{R}$ *is an open interval). Any solution to* $\dot{x} = v(x,t)$ *is smooth.*

Proof. Let ϕ be a solution. Since $\dot{\phi}$ exists, then ϕ is differentiable, and thus continuous. Since v is continuous as well, $\dot{\phi} = v(\phi, t)$ is continuous and so ϕ is (locally) C^1. Now since v is smooth, we may differentiate to find

$$\ddot{\phi}(t) = \frac{\partial v}{\partial x^i}(\phi, t)\dot{\phi}^i(t) + \frac{\partial v}{\partial t}(\phi, t).$$

Now since ϕ and $\dot{\phi}$ and the partial derivatives of v are continuous, we see that $\ddot{\phi}$ is continuous and ϕ is (locally) C^2. Since v is smooth, we can keep differentiating, using the chain and product rules, to find by induction $d^m\phi/dt^m$ is continuous for all m and so ϕ is C^∞. □

Remark. The technique used in the proof of Theorem 9 above is called *bootstrapping.* In this process, once we know that ϕ is C^0, we plug into the equation to find that ϕ is C^1. Then we use the fact that ϕ is C^1 to prove ϕ is C^2, etc.

Remark. The proof above also shows that if v is C^k, then ϕ is C^{k+1}.

2.6 Higher order equations

A higher-order systems of ODEs is of the form

$$x^{(m)} = v(x^{(m-1)}, \dots, \dot{x}, x, t), \tag{17}$$

where of course $x^{(m)} = \frac{d^m x}{dt^m}$. There is an easy trick to transform this system to an equivalent first-order system with more variables. Let $y_1 = \dot{x}, \dots, y_{m-1} = x^{(m-1)}$. Then it is easy to see the system (17) above is equivalent to the system

$$\left\{ \begin{array}{rcl} \dot{y}^{m-1} & = & v(y^{m-1}, \dots, y^1, x, t), \\ \dot{y}^{m-2} & = & y^{m-1}, \\ \vdots & & \vdots \\ \dot{y}^1 & = & y^2, \\ \dot{x} & = & y^1. \end{array} \right. \tag{18}$$

This first-order system leads us to the appropriate formulation of the initial-value problem:

Theorem 10. *Let $\mathcal{O}$ be a neighborhood of $(x_0^{m-1}, \dots x_0^1, x_0)$ in $\mathbb{R}^{nm} = \mathbb{R}^n \times \dots \times \mathbb{R}^n$, and let I be an open interval containing t_0. Let $v \colon \mathcal{O} \times I \to \mathbb{R}^n$ be locally Lipschitz. Then there is an interval $J \subset I$ on which there is a unique solution to the initial value problem*

$$\left\{ \begin{array}{rcl} x^{(m)} & = & v(x^{(m-1)}, \dots, \dot{x}, x, t), \\ x^{(m-1)}(t_0) & = & x_0^{m-1}, \\ \vdots & & \vdots \\ \dot{x}(t_0) & = & x_0^1, \\ x(t_0) & = & x_0. \end{array} \right. \tag{19}$$

Moreover, if there is a point $T \in (\partial J) \cap I$, then as $t \to T$, $(x^{(m-1)}, \dots, \dot{x}, x)$ leaves every compact subset of $\mathcal{O}$.

Proof. Apply Theorems 5 and 7. ☐

So for an m^{th} order differential equation, we need initial conditions for the function and its derivatives up to order $m-1$.

Remark. The trick of introducing new variables into a system of ODEs is standard in physics. For a particle at position $x = x(t)$, a typical equation involves how a force acts on the particle. The sum F of the forces acting on the particle must be equal to $m\ddot{x}$, where m is a constant called the mass. It is standard to introduce a new vector quantity, called the *momentum* $q = m\dot{x}$. Then $F = m\ddot{x}$ is equivalent to the system

$$\dot{q} = F, \qquad \dot{x} = \frac{q}{m}.$$

Again, an important class of examples is linear equations with constant coefficients. If

$$x^{(m)} + a_{m-1}x^{(m-1)} + \cdots + a_1\dot{x} + a_0 x = 0,$$

for x a real-valued function, the functions $\{e^{\lambda_k t}\}$ are linearly independent in the solution space, if λ_k solve the *characteristic equation*

$$\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0.$$

If all the roots are distinct, then $\{e^{\lambda_k t}\}$ form a basis. If a root is repeated l times, then we must consider functions of the form $t^j e^{\lambda_k t}$ for $j = 0, \ldots, l-1$ to form a basis of the solution space.

Euler's formula again allows us to handle complex roots of the characteristic equation.

Homework Problem 20. *For which real values of the constants a and b do all the solutions to*

$$\ddot{x} + a\dot{x} + bx = 0$$

go to 0 as $t \to \infty$? Prove your answer, and draw your answer as a region in the (a, b) plane.

2.7 Dependence on initial conditions and parameters

We've shown above that if $v = v(x,t)$ is smooth, then the resulting solution to $\dot{x} = v(x,t), x(t_0) = x_0$ is also smooth as a function of t. The initial value problem also depends on the initial point x_0. We investigate regularity of the solution depending on x_0.

First of all we remark that there is a neighborhood $\mathcal{N}$ of (x_0, t_0) in $\mathbb{R}^{n+1}$ and an $\epsilon > 0$ so that every solution to the equation with initial condition $x(\tau) = y$ for $(y, \tau) \in \mathcal{N}$ exists by Lemma 23. This existence on a neighborhood allows us to consider taking derivatives in y in what follows.

Theorem 11. *Let v be a C^2 function on a neighborhood of the initial conditions $(y, t_0) \in \mathbb{R}^n \times \mathbb{R}$. Then the solution $\phi = \phi(y, t)$ to the initial value problem*

$$\dot{x} = v(x, t), \qquad x(t_0) = y,$$

is C^1 in y.

Proof. If $\partial\phi/\partial y^i$ exists, then it must satisfy

$$\frac{\partial}{\partial t} D_y\phi = D_x v(\phi, t) \circ D_y\phi.$$

(Here $D_y\phi$ is the total derivative matrix with respect to the y variables. So its entries are $\partial\phi^j/\partial y^i$.) So $\Phi = (\phi, D_y\phi) = (x, z)$ should satisfy the initial value problem

$$\left\{\begin{array}{rcl} \dot{x} &=& v(x,t), \\ \dot{z} &=& D_x v(x,t) \circ z, \\ x(t_0) &=& y, \\ z(t_0) &=& I \qquad \text{the identity matrix.} \end{array}\right. \tag{20}$$

Note that since v is C^2, $D_x v = (\partial v^k/\partial x^j)$ is C^1 and is thus locally Lipschitz by Proposition 15. Even though we don't yet know that the derivative $D_y\phi$ satisfies the equation, we do know that the initial value problem (20) is solvable.

In order show the solution to (20) is the partial derivative, we return to the proof of Theorem 5. Let $\phi_0 = y$, $\psi_0 = I$ the identity matrix. Then (ϕ_0, ψ_0) satisfy the initial conditions in (20). Now we form Picard approximations

$$\begin{aligned} \phi_{n+1}(y,t) &= y + \int_{t_0}^{t} v(\phi_n(y,\tau), \tau)\, d\tau, \\ \psi_{n+1}(y,t) &= I + \int_{t_0}^{t} D_x v(\phi_n(y,\tau), \tau) \circ \psi_n(y,\tau)\, d\tau. \end{aligned}$$

It is easy to show by induction that $D_y\phi_n = \psi_n$. We already have the initial step $n = 0$, and since we can differentiate under the integral sign (see Proposition 11 above), we can easily check that $D_y\phi_n = \psi_n$ implies $D_y\phi_{n+1} = \psi_{n+1}$.

We know by the proof of Theorem 5 that $\phi_n \to \phi$ and $\psi_n \to \psi$ uniformly on a small interval containing t_0. Then Proposition 8 shows that $\partial\phi/\partial y^i = \psi_i$ the i^{th} component of ψ for $i = 1, \dots, n$. Since these partial derivatives are continuous (the uniform limit of continuous functions is continuous), then Proposition 6 shows $D_y\phi = \psi$. □

Remark. The previous theorem is true if we assume v is only C^1 and not necessarily C^2. The proof is more involved in the case v is only C^1. (See Taylor, *Partial Differential Equations, Basic Theory*, section 1.6.)

A bootstrapping argument can be used to prove the following

Proposition 29. *For $r \geq 2$, let v be a C^r function on a neighborhood of the initial conditions $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$. Then the solution $\phi = \phi(y, t)$ to the initial value problem*

$$\dot{x} = v(x, t), \qquad x(t_0) = y,$$

is C^{r-1} in y.

Proof. Let Proposition T_r be the proposition for a given $r \geq 2$. We proceed by induction. The case $r = 2$ is proved above in Theorem 11. Now assume that the Proposition T_r has been proved. To prove T_{r+1}, assume that v is locally C^{r+1} and let ϕ be a solution to the initial value problem. Then $D_x v$ is locally C^r. Now as above, the pair $(\phi, D_y\phi) = (x, z)$ satisfies

$$\dot{x} = v(x, t), \qquad \dot{z} = D_x v(x, t) \circ z. \tag{21}$$

Now analyze the right-hand side of the equations in (21). They are C^r functions of x, z, t. Therefore, Proposition T_r shows that $z = D_y\phi$ is locally C^{r-1} in y. Since the first partial derivatives of ϕ are C^{r-1}, ϕ is C^r. This proves the inductive step, and the proposition. □

We also have the following

Corollary 30. *If $v = v(x, t)$ is smooth (C^∞), then the solution ϕ to the initial value problem $\dot{x} = v(x, t)$, $x(t_0) = y$ is smooth in y.*

Moreover, it is not too hard to prove the following:

Theorem 12. *Let $r \geq 2$. If $v(x,t)$ is C^r jointly in x and t, and if ϕ is the solution to $\dot{x} = v(x,t)$, $x(t_0) = y$, then ϕ is jointly C^{r-1} in y, t and t_0.*

Idea of proof. The difficult part is already done (the C^{r-1} dependence on y). For the rest, recall that any solution $\phi = \phi(y, t_0, t)$ satisfies

$$\phi = y + \int_{t_0}^{t} v(\phi(y, t_0, \tau), \tau)\, d\tau.$$

Then use the Fundamental Theorem of Calculus and Proposition 11 below to produce a bootstrapping argument to show that the appropriate partial derivatives are continuous.

For a complete proof, see Arnol'd, *Ordinary Differential Equations*, section 32.5. $\square$

Homework Problem 21. *For $f = f(x,t,y)$ a smooth function real variables of x, t, and y, compute*

$$\frac{d}{dt} \int_0^{t^2} f(x(t,y), t, y)\, dy.$$

Make sure your answer works for the functions $f(x,t,y) = x^2 ty + t^3 y^2 + x$, $x(t,y) = y^2 + t^2$.

Hint: Carefully rename all intermediate variables and apply the Chain Rule. It also should help to write down the anti-derivative $F = \int f(x(t,y), t, y)\, dy$ and work with the function F using the Fundamental Theorem of Calculus.

Homework Problem 22 (Smooth dependence on parameters). *Show that if $v = v(x,t,\alpha)$ is jointly smooth on a neighborhood of (x_0, t_0, α_0) in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$, then the solution ϕ to the initial value problem*

$$\dot{x} = v(x,t,\alpha), \qquad x(t_0) = x_0$$

is smooth as a function of α.

Hint: Show that this initial value problem is equivalent to the problem

$$\dot{x} = v(x,t,\beta), \quad x(t_0) = x_0, \quad \dot{\beta} = 0, \quad \beta(t_0) = \alpha.$$

2.8 Autonomous equations

An ODE system of the form $\dot{x} = v(x)$ is *autonomous*. In other words, a system is autonomous if there is no explicit dependence on t. The main fact about autonomous systems is the following proposition, whose proof is an easy computation:

Proposition 31. *If ϕ is a solution to $\dot{x} = v(x)$, then for all $T \in \mathbb{R}$, $\tilde{\phi}(t) = \phi(t+T)$ is also a solution.*

A constant solution to an ODE system is called an *equilibrium solution*. The equilibrium solutions to autonomous equations correspond to the roots of v.

Example 9. *Consider the initial value problem $\dot{x} = x^2 - 1$. Then to solve, we have the equilibrium solutions $x = 1$ and $x = -1$. If $x^2 - 1 \neq 0$, compute*

$$\begin{aligned}
\frac{dx}{dt} &= x^2 - 1, \\
\int \frac{dx}{x^2 - 1} &= \int dt, \\
\int \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx &= t + C, \\
\frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| &= t + C, \\
\frac{x-1}{x+1} &= \pm e^{2t+2C} \\
&= Ae^{2t}, \quad A = \pm e^{2C} \neq 0, \\
x &= \frac{1 + Ae^{2t}}{1 - Ae^{2t}}, \\
A &= \frac{x(0) - 1}{x(0) + 1}.
\end{aligned}$$

If $x(0) \in (-1, 1)$, then $A < 0$, and the solution x exists for all time and is bounded between the equilibrium solutions at 1 and -1. Moreover, x approaches the equilibrium solutions $x \to -1$ as $t \to \infty$ and $x \to 1$ as $t \to -\infty$. If $x(0) > 1$, then $A \in (0, 1)$ and the solution exists only for $t \in (-\infty, -\frac{1}{2} \ln A)$. If $x(0) < -1$, then $A > 1$ and the solution exists only for $t \in (-\frac{1}{2} \ln A, \infty)$.

This behavior is typical of the behavior of autonomous equations for Lipschitz v. Any bounded solution which exists for all time must be asymptotic to equilibrium solutions as $t \to \pm\infty$. Also note that any integral curve $\mathcal{I}$ acts as a *barrier* to other solutions, in that no other integral curves can cross $\mathcal{I}$ (see Proposition 26 above).

Homework Problem 23. *Let* $v : \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz. Show that any bounded solution* ϕ *of* $\dot{x} = v(x)$ *which exists for all time satisfies* $\lim_{t\to\infty} \phi(t) = c$*, where* $v(c) = 0$*.*

Hint: There are three cases:
Case 1: $v(\phi(0)) = 0$*. Show that* ϕ *is constant by uniqueness.*
Case 2: $v(\phi(0)) > 0$*. Show that* $v(\phi(t)) > 0$ *for all* t *(if it is ever equal to zero, apply the argument of Case 1 above to show* ϕ *is constant; also use the continuity of* $v \circ \phi$*). Now show* $\phi(t)$ *is always increasing, and so must have a finite limit* c *as* $t \to \infty$*. Compute* $\lim_{t\to\infty} v(\phi(t))$*. Write*

$$\infty > c = \phi(0) + \int_0^\infty \dot{\phi}(t)\,dt = \phi(0) + \int_0^\infty v(\phi(t))\,dt,$$

and show that $v(c) = 0$*.*
Case 3: $v(\phi(0)) < 0$ *is essentially the same as Case 2.*

2.9 Vector fields and flows

An important interpretation of autonomous systems of equations is given in terms of vector fields. Interpret $x(t)$ as a parametrized curve $x : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval. Then $\dot{x}(t)$ is the *tangent vector* to the curve at time t. For $\mathcal{O} \subset \mathbb{R}^n$ an open set, a function $v : \mathcal{O} \to \mathbb{R}^n$ can be thought of as a *vector field*. In other words, at every point $x \in \mathcal{O}$, $v(x)$ is a vector in $\mathbb{R}^n$ based at x. Then we have a natural interpretation of an autonomous differential equation $\dot{x} = v(x)$ as the *flow* along the vector field v.

For any solution to $\dot{x} = v(x)$, the tangent vector $\dot{x}(t)$ must be equal to the value of the vector field $v(x(t))$. The solution $x(t)$ is an *integral curve* to the equation $\dot{x} = v(x)$. The integral curves for the solution are tangent to the vector field at each point x. Moreover, if $v(x)$ is locally Lipschitz, then the solutions are unique, and we may think of the vector field as giving unique directions for how to proceed in time at each point in space. By the invariance of solutions in time, we have the following strong version of uniqueness:

Proposition 32. *Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set, and let $v\colon \mathcal{O} \to \mathbb{R}^n$ be locally Lipschitz. If ϕ_1 and ϕ_2 are two maximally extended solutions to $\dot{x} = v(x)$ which satisfy $\phi_1(t_1) = \phi_2(t_2)$, then $\phi_1(t) = \phi_2(t + t_2 - t_1)$ for all t in the maximal interval of definition of ϕ_1.*

Proof. $\phi_1(t)$ and $\tilde{\phi}_2(t) = \phi_2(t + t_2 - t_1)$ both satisfy the initial value problem

$$\dot{x} = v(x), \qquad x(t_1) = \phi_1(t_1),$$

and so must be the same by Theorems 5 and 6. □

For a vector field v on $\mathcal{O} \subset \mathbb{R}^n$, a picture of all the integral curves on $\mathcal{O}$ is called the *phase portrait* of v. Recall we drew in class the phase portraits of the two systems in $\mathbb{R}^2$

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x, \qquad \dot{x} = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} x.$$

Homework Problem 24.

(a) Draw the phase portrait of the system in $\mathbb{R}^2$

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x.$$

Show that each integral curve lies in a parabola or a line in $\mathbb{R}^2$.

(b) Draw the phase portrait of the system in $\mathbb{R}^2$

$$\dot{x} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} x.$$

Here is the principal theorem regarding flows of vector fields on open sets:

Theorem 13. *Let $\mathcal{O} \subset \mathbb{R}^n$ be open, and $v\colon \mathcal{O} \to \mathbb{R}^n$ be smooth. Then there is an open set $\mathcal{U}$ so that $\mathcal{O} \times \{0\} \subset \mathcal{U} \subset \mathcal{O} \times \mathbb{R}$ on which the solution $\phi(y, t)$ to*

$$\dot{x} = v(x), \qquad x(0) = y$$

exists, is unique, and is smooth jointly as a function of (y, t).

Proof. This follows immediately from Theorems 5, 7 and 11. □

Remark. It may not be possible to find an $\epsilon > 0$ so that $\mathcal{O} \times (-\epsilon, \epsilon) \subset \mathcal{U}$. The reason is that solutions may leave $\mathcal{O}$ in shorter and shorter times for initial conditions $y \to \partial\mathcal{O}$. A simple example is given by $v(x) = 1$, $\mathcal{O} = (0,1)$. This problem cannot be fixed by considering $\mathcal{O} = \mathbb{R}^n$, since we may have $v(y) \to \infty$ rapidly as $y \to \infty$ in $\mathbb{R}^n$. However, see the following corollary.

Corollary 33. *Under the conditions of Theorem 13 above, if $K \subset \mathcal{O}$ is compact, then there is an $\epsilon > 0$ so that the solution*

$$\phi\colon K \times (-\epsilon, \epsilon) \to \mathcal{O}.$$

Proposition 34. *Consider $\phi(y,t)$ the solution to $\dot{x} = v(x)$, $x(0) = y$, for v smooth. Then as long as $\phi(y,t_1), \phi(y,t_1+t_2) \in \mathcal{O}$, then*

$$\phi(y, t_1 + t_2) = \phi(\phi(y,t_1), t_2).$$

Proof. Consider

$$\psi(t) = \phi(y, t_1 + t), \qquad \theta(t) = \phi(\phi(y,t_1), t).$$

Then if we show ψ and θ satisfy the same initial value problem, then uniqueness will show that $\psi(t) = \theta(t)$ and we are done.

Compute

$$\begin{aligned}
\psi(0) &= \phi(y,t_1), \\
\theta(0) &= \phi(\phi(y,t_1),0) = \phi(y,t_1), \\
\dot{\psi}(t) &= \dot{\phi}(y, t_1 + t) \cdot 1 = v(\phi(y, t_1 + t)) = v(\psi(t)), \\
\dot{\theta}(t) &= \dot{\phi}(\phi(y,t_1),t) = v(\phi(\phi(y,t_1),t)) = v(\theta(t)).
\end{aligned}$$

□

Note that it is necessary in the previous Proposition 34 to restrict to times in which the solution does not leave $\mathcal{O}$. In fact, long-time existence of flows along vector fields is problematic on open subsets of $\mathbb{R}^n$. Recall we require our subsets to be open for ODEs since we want to be able to take two-sided limits for any derivatives involved. On the other hand, compactness guarantees a uniform time interval for existence. But compact subsets of $\mathbb{R}^n$ are closed and bounded, and thus (if nonempty) cannot be open. The way out of this problem is to consider compact manifolds, which we will realize as compact lower-dimensional subsets of $\mathbb{R}^n$. For example,

$$\mathbb{S}^1 = \{(x^1, x^2) : (x^1)^2 + (x^2)^2 = 1\}$$

is a compact one-dimensional submanifold of $\mathbb{R}^2$.

2.10 Vector fields as differential operators

A vector field v on $\mathcal{O}$ naturally differentiates functions f on $\mathcal{O}$ by the directional derivative:

$$vf = \mathcal{D}_v f = v^i \frac{\partial f}{\partial x^i}$$

for v^i the components of v. Therefore, we often write

$$v = v^i \frac{\partial}{\partial x^i}.$$

We say that v is a *first-order differential operator* on functions f.

This observation is natural from the point of view of ODEs by the following

Proposition 35. *For an interval $I \subset \mathbb{R}$, let $\phi\colon I \to \mathbb{R}^n$ be a solution to the autonomous system $\dot{x} = v(x)$, where $v\colon \mathcal{O} \to \mathbb{R}^n$ is a continuous function and $\mathcal{O}$ an open subset in $\mathbb{R}^n$. Also consider a differentiable function $f : \mathcal{O} \to \mathbb{R}$. Then the derivative*

$$(f \circ \phi)'(t) = (D_v f)(\phi(t)) = (vf)(\phi(t)).$$

Proof. Compute

$$\begin{aligned}
(f \circ \phi)'(t) &= (Df)(\phi(t)) \circ (D\phi)(t) \\
&= \frac{\partial f}{\partial x^i}(\phi(t)) \frac{d\phi^i}{dt}(t) \\
&= \frac{\partial f}{\partial x^i}(\phi(t)) v^i(\phi(t)) \\
&= \left(v^i \frac{\partial}{\partial x^i} f \right)(\phi(t)) \\
&= (vf)(\phi(t)).
\end{aligned}$$

□

Define the bracket $[v, w]$ of two operators to be

$$[v, w]f = (vw - wv)f = v(wf) - w(vf).$$

Homework Problem 25. *Let v and w are two smooth vector fields on Ω.*

(a) *Show that the differential operator* $[v, w]$ *is also a first-order differential operator determined by a vector field (which we also write as* $[v, w]$*). What are the components of* $[v, w]$*?*

(b) *For smooth vector fields* u, v *and* w, *show that*

$$[u, v] = -[v, u]$$

and

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0.$$

(This last identity is the Jacobi identity.)

Remark. Part (b) of the previous problem shows that the vector space of smooth vector fields on $\mathcal{O}$ is a *Lie algebra.* The bracket $[\cdot, \cdot]$ is called the *Lie bracket.*