3 Manifolds

3.1 Smooth manifolds

We define smooth manifolds as subsets of $\mathbb{R}^N$. We basically follow Spivak, *Calculus on Manifolds*, Chapter 5. When we say smooth in this section, we mean C^∞.

We say a subset $M \subset \mathbb{R}^n$ is a smooth k-dimensional *manifold* (or, more properly, a *submanifold* of $\mathbb{R}^n$), if for all $x \in M$, there are open subsets $\mathcal{U} \subset \mathbb{R}^k$ and $\mathcal{O} \subset M$ and a one-to-one C^∞ map $\phi\colon \mathcal{U} \to \mathbb{R}^n$ satisfying

1. $\phi(\mathcal{U}) = \mathcal{O}$.
2. For all $y \in \mathcal{U}$, $D\phi(y)$ has rank k.
3. $\phi^{-1}\colon \mathcal{O} \to \mathcal{U}$ is continuous.

Such a pair $(\phi,\mathcal{U})$ is called a *local parametrization* of M. The components of the map $\phi^{-1} : \mathcal{O} \to \mathbb{R}^k$ are *local coordinates* on M. A set of triples $(\phi_\alpha,\mathcal{U}_\alpha,\mathcal{O}_\alpha)$ is called an *atlas* of M if $\{\mathcal{O}_\alpha\}$ is an open cover of M.

Since $\mathcal{O}$ is an open subset of M, there is an open subset $W \subset \mathbb{R}^n$ so that $\mathcal{O} = M \cap W$. In this case, we may rewrite condition (1) as

(1′) $\phi(\mathcal{U}) = M \cap W$.

Also note that $\phi : \mathcal{U} \to \mathcal{O}$ is a homeomorphism from $\mathcal{O}$ to $\mathcal{U}$ since it is smooth, one-to-one, onto, and ϕ^{-1} is continuous.

Now we note with a few examples why conditions (2) and (3) are necessary. First of all, consider $\phi : \mathbb{R} \to \mathbb{R}^2$ given by $\phi(t) = (t^2, t^3)$. Then ϕ is smooth, one-to-one, and $\phi^{-1} : \phi(\mathbb{R}) \to \mathbb{R}$ is continuous. But we note the image $\phi(\mathbb{R})$, which is the graph of $x^1 = (x^2)^{\frac{2}{3}}$ in $\mathbb{R}^2$, is not smooth at $(0,0) \in \mathbb{R}^2$. We also check that

$$D\phi = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} = 0 \qquad \text{when } t = 0 \text{ and } \phi(t) = (0,0),$$

and so $D\phi$ has rank $0 < 1$ at the point at which $\phi(\mathbb{R})$ is not smooth.

Condition (3) is necessary by the following problem:

Homework Problem 26. *Recall polar coordinates* $(x, y) = (r\cos\theta, r\sin\theta)$ *in* $\mathbb{R}^2$. *Show that a portion of the polar graph* $r = \sin 2\theta$ *can be parametrized* *for I an open interval in $\mathbb{R}$, by $\phi\colon I \to \mathbb{R}^2$ so that ϕ is one-to-one, C^∞, and $D\phi$ is never 0, but so that $\phi^{-1}\colon \phi(I) \to I$ is not continuous. Sketch the graph and indicate pictorially why $\phi(I)$ should not be considered a submanifold of $\mathbb{R}^2$.*

If W and V are open subset of $\mathbb{R}^n$, then a map $f\colon W \to V$ is a *diffeomorphism* if f is one-to-one, onto, C^∞, and f^{-1} is C^∞. The Inverse Function Theorem and Problem 9 show

Lemma 36. *$f\colon W \to V$ is a diffeomorphism if and only if f is one-to-one, onto, C^∞, and $\det Df(x) \neq 0$ for all $x \in W$.*

The following theorem is useful in proving properties about manifolds:

Theorem 14. *$M \subset \mathbb{R}^n$ is a k-dimensional manifold if and only if for all $x \in M$, there are two open subset V, W of $\mathbb{R}^n$, with $x \in W$ and a diffeomorphism $h\colon W \to V$ satisfying*

$$h(W \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V : y^{k+1} = \cdots = y^n = 0\}.$$

Proof. ($\Leftarrow$) Let $\mathcal{U} = \{a \in \mathbb{R}^k : (a, 0) \in h(W)\}$, and define $\phi : \mathcal{U} \to \mathbb{R}^n$ by $\phi(a) = h^{-1}(a, 0)$. ϕ is smooth and one-to-one since h is a diffeomorphism. Moreover, $\phi(\mathcal{U}) = M \cap W$ to satisfy condition ($1'$). $\phi^{-1} = h\big|_{(W\cap M)}$ is continuous.

So all that is left to check is the rank condition (2). Consider $H\colon W \to \mathbb{R}^k$

$$H(z) = (h^1(z), \ldots, h^k(z)).$$

Then $H(\phi(y)) = y$ for all $y \in \mathcal{U}$. Then use the Chain Rule to compute $DH(\phi(y)) \circ D\phi(y) = I$, and so $D\phi(y)$ must be an injective linear map, and so must have rank k. Thus M is a smooth manifold.

($\Rightarrow$) Now assume M is a manifold, and define $y = \phi^{-1}(x)$. Then $D\phi(y)$ has rank k, and so there is at least on $k \times k$ submatrix of $D\phi(y)$ with nonzero determinant. (We may think of $D\phi(y)$ as an $n \times k$ matrix mapping column vectors in $\mathbb{R}^k$ to column vectors in $\mathbb{R}^n$. Then a $k \times k$ submatrix is simply a collection of k distinct rows of $D\phi(y)$.) By a linear change of basis, if necessary, then, we may assume that

$$\det_{1 \le i,j \le k} \left(\frac{\partial \phi^i}{\partial y^j} \right)(y) \neq 0.$$

By continuity, this is true on an open neighborhood $\mathcal{U}'$ of y.

Define $g\colon \mathcal{U}' \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by $g(a,b) = \phi(a) + (o,b)$. Then, in block matrix form,

$$Dg(a,b) = \begin{pmatrix} \left(\dfrac{\partial \phi^i}{\partial y^j}\right)_{1\le i,j\le k} & 0 \\ \left(\dfrac{\partial \phi^i}{\partial y^j}\right)_{1\le j\le k, k<i\le n} & I_{n-k} \end{pmatrix}.$$

So $\det Dg(a,b) = \det_{1\le i,j\le k}\left(\frac{\partial \phi^i}{\partial y^j}\right) \neq 0$. So we may apply the Inverse Function Theorem to find that there are open subsets of $\mathbb{R}^n$ $V_1' \ni (y,0)$ and $V_2' \in g(y,0) = x$ so that $g\colon V_1' \to V_2'$ has a smooth inverse $h\colon V_2' \to V_1'$.

Define $\mathcal{O}$ via

$$\begin{aligned} \mathcal{O} &= \{\phi(a) : (a,0) \in V_1'\} \\ &= (\phi^{-1})^{-1}(\iota^{-1}(V_1')), \end{aligned}$$

where $\iota : \mathbb{R}^k \to \mathbb{R}^n$ sends a to $(a,0)$. Since ϕ^{-1} is continuous, $\mathcal{O}$ is an open subset of $\phi(\mathcal{U}')$, and of M. Therefore, there is an open subset $\tilde{V}$ of $\mathbb{R}^n$ so that $\mathcal{O} = M \cap \tilde{V}$.

Let $W = \tilde{V} \cap V_2$, and $V = g^{-1}(W)$. Then $h\colon V \to W$ is a diffeomorphism and

$$\begin{aligned} W \cap M &= \{\phi(a) : (a,0) \in V\} \\ &= \{g(a,0) : (a,0) \in V\}, \\ h(W \cap M) &= g^{-1}(W \cap M) \\ &= g^{-1}\left(\{g(a,0) : (a,0) \in V\}\right) \\ &= V \cap (\mathbb{R}^k \times \{0\}). \end{aligned}$$

This completes the proof. □

This characterization of manifolds is quite useful. Consider two smooth local parametrizations $\phi_\alpha\colon \mathcal{U}_\alpha \to \mathcal{O}_\alpha$, and $\phi_\beta\colon \mathcal{U}_\beta \to \mathcal{O}_\beta$. Then if $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$, then we have the following

Proposition 37. *$\phi_\beta^{-1} \circ \phi_\alpha\colon \phi_\alpha^{-1}(\mathcal{O}_\beta) \to \phi_\beta^{-1}(\mathcal{O}_\alpha)$ is a diffeomorphism.*

Proof. Consider $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ given by $(a,b) \mapsto a$ for $(a,b) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, and $\iota : \mathbb{R}^k \to \mathbb{R}^n$ given by $\iota(a) = (a,0)$. Let h_α and h_β be the diffeomorphisms guaranteed by Theorem 14. Then $\phi_\alpha(a) = h_\alpha^{-1}(a, 0)$, $\phi_\alpha^{-1}(x) = \pi(h_\alpha(x))$, and so

$$\phi_\beta^{-1} \circ \phi_\alpha = \pi \circ h_\beta \circ h_\alpha^{-1} \circ \iota$$

is smooth since h_α, h_β are diffeomorphisms. □

The maps $\phi_\beta^{-1} \circ \phi_\alpha$ are called *gluing maps.*

Remark. It is often useful to think of a manifold M as being glued together from domains $\mathcal{U}_\alpha$ in $\mathbb{R}^k$ by the gluing maps. In fact, the previous proposition is the starting point for the abstract definition of a smooth manifold: A smooth k-dimensional manifold is Hausdorff, sigma-compact topological space for which each point x has a neighborhood $\mathcal{O}_\alpha$ homeomorphic to a domain $\mathcal{U}_\alpha \subset \mathbb{R}^k$ via $\phi_\alpha : \mathcal{U}_\alpha \to \mathcal{O}_\alpha$. In addition, we require the gluing maps $\phi_\beta^{-1} \circ \phi_\alpha$ to be smooth on $\phi_\alpha^{-1}(\mathcal{O}_\beta)$.

If M is a smooth manifold, then a function $f : M \to \mathbb{R}^p$ is said to be smooth if for each smooth parametrization $\phi : \mathcal{U} \to M$, $f \circ \phi : \mathcal{U} \to \mathbb{R}^p$ is smooth. If $N \subset \mathbb{R}^p$ is a smooth submanifold, then $f : M \to N$ is said to be smooth the induced map $f : M \to \mathbb{R}^p$ is smooth. (For abstract target manifolds N, we may work with local parametrizations instead.) This definition of smooth maps from manifolds is consistent in the following sense:

Proposition 38. *If $f : M \to \mathbb{R}^p$, and $f \circ \phi_\alpha$ is smooth from $\mathcal{U}_\alpha \to \mathbb{R}^p$, then on $\phi_\beta^{-1}(\mathcal{O}_\alpha) \subset \mathcal{U}_\beta$, $f \circ \phi_\beta$ is also smooth.*

Proof. Apply Proposition 37 and the Chain Rule. □

Proposition 39. *If $M \subset \mathbb{R}^n$ is a smooth manifold and $f : M \to \mathbb{R}^p$, then f is smooth if and only if f can be locally extended to smooth functions from domains in $\mathbb{R}^n$ to $\mathbb{R}^p$. In other words, f is smooth if and only if every $x \in M$ has a neighborhood W in $\mathbb{R}^n$, and there is a smooth function $F : W \to \mathbb{R}^p$ so that $F\big|_{W \cap M} = f$.*

Proof. ($\Rightarrow$) For $x \in M$, consider the local diffeomorphism $h : W \to V$ guaranteed by Theorem 14. Then for the smooth parametrization $\phi(a) = h^{-1}(a, 0)$, we know $f \circ \phi$ is smooth. Now define

$$F = f \circ h^{-1} \circ \pi \circ h : W \to \mathbb{R}^p$$

for $\pi : (a, b) \mapsto a$. F is smooth since

$$F = f \circ h^{-1} \circ \pi \circ h = (f \circ \phi) \circ \pi \circ h.$$

(⇐) For a local parametrization ϕ, $f \circ \phi$ is smooth since locally, $f \circ \phi = F \circ \phi$, which is smooth by the Chain Rule. □

$X \subset \mathbb{R}^N$ is a smooth manifold of dimension k if every $x \in X$ has a neighborhood that is diffeomorphic to an open subset of $\mathbb{R}^k$. In other words, there is an open cover $\mathcal{O}_\alpha$ of X so that each $\mathcal{O}_\alpha$ is diffeomorphic to an open subset $\mathcal{U}_\alpha \subset \mathbb{R}^k$. Let $\phi_\alpha : \mathcal{U}_\alpha \to \mathcal{O}_\alpha$ be the diffeomorphism. ϕ_α is called a *parametrization* of $\mathcal{O}_\alpha \subset X$, and the inverse map ϕ_α^{-1} is called a *coordinate system*. The open cover, together with the coordinate systems

$$\{\mathcal{O}_\alpha, \phi_\alpha, \mathcal{U}_\alpha\}$$

is called a smooth *atlas* of X, and X is a smooth manifold if and only if it has a smooth atlas.

Example 10. *The unit sphere*

$$\mathbb{S}^2 = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$$

is a two-dimensional submanifold of $\mathbb{R}^3$.

To show this, we provide an atlas. Let $N = (0, 0, 1)$ *be the north pole and* $S = (0, 0, -1)$ *be the south pole. Then let* $\mathcal{O}_1 = \mathbb{S}^2 \setminus \{N\}$, $\mathcal{O}_2 = \mathbb{S}^2 \setminus \{S\}$, $\mathcal{U}_1 = \mathcal{U}_2 = \mathbb{R}^2$. *We construct the coordinate systems* ϕ_α^{-1}, $\alpha = 1, 2$, *by* stereographic projection. *We may realize* $\mathbb{R}^2$ *as the plane* $\{x^3 = 0\} \subset \mathbb{R}^3$.

For a point x *in* $\mathcal{O}_1$, *consider the line* $L_{x,N}$ *in* $\mathbb{R}^3$ *through* N *and* x. *We define* $\phi_1^{-1}(x)$ *to be the unique point in* $\mathbb{R}^2 \cap L_{x,N}$. *It is easy to compute*

$$\begin{aligned}(y^1, y^2) = \phi_1^{-1}(x^1, x^2, x^3) &= \left(\frac{x^1}{1 - x^3}, \frac{x^2}{1 - x^3}\right),\\ (x^1, x^2, x^3) = \phi_1(y^1, y^2) &= \left(\frac{2y^1}{|y|^2 + 1}, \frac{2y^2}{|y|^2 + 1}, \frac{|y|^2 - 1}{|y|^2 + 1}\right).\end{aligned}$$

Similarly, for any point $x \in \mathcal{O}_2$, *define* $\phi_2^{-1}(x)$ *to be the unique point in* $\mathbb{R}^2 \cap L_{x,S}$, *and we find as above*

$$\begin{aligned}(z^1, z^2) = \phi_2^{-1}(x^1, x^2, x^3) &= \left(\frac{x^1}{1 + x^3}, \frac{x^2}{1 + x^3}\right),\\ (x^1, x^2, x^3) = \phi_2(z^1, z^2) &= \left(\frac{2z^1}{|z|^2 + 1}, \frac{2z^2}{|z|^2 + 1}, -\frac{|z|^2 - 1}{|z|^2 + 1}\right).\end{aligned}$$

It is straightforward to check that each of these coordinate systems is a diffeomorphism, and since $\mathbb{S}^2 = \mathcal{O}_1 \cup \mathcal{O}_2$, *we have produced a smooth atlas of* $\mathbb{S}^2$ *and thus have shown that* $\mathbb{S}^2$ *is a two-dimensional manifold.*

Given a smooth manifold X with a smooth atlas $\{\mathcal{O}_\alpha, \phi_\alpha, \mathcal{U}_\alpha\}$, let $\mathcal{O}_{\alpha\beta} = \mathcal{O}_\alpha \cap \mathcal{O}_\beta$. Also define $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \phi_\alpha^{-1}(\mathcal{O}_{\alpha\beta})$. As long as $\mathcal{O}_{\alpha\beta} \neq \emptyset$, the map

$$\phi_{\alpha\beta} \equiv \phi_\beta^{-1} \circ \phi_\alpha : \mathcal{U}_{\alpha\beta} \to \mathcal{U}_{\beta\alpha}$$

is a diffeomorphism. These maps $\phi_{\alpha\beta}$ are called the *gluing maps* of the manifold X associated to the atlas. In particular, the manifold can be thought of as the union of the coordinate charts U_α glued together by the gluing maps. It is straightforward to see, at least as a set, we may identify

$$X = \left(\bigsqcup_\alpha \mathcal{U}_\alpha \right) / \sim,$$

where $\sqcup$ means disjoint union and the equivalence relation $\sim$ is given by

$$x \sim y \quad \text{if } x \in \mathcal{U}_{\alpha\beta} \subset \mathcal{U}_\alpha,\ y \in \mathcal{U}_{\beta\alpha} \subset \mathcal{U}_\beta,\ y = \phi_{\alpha\beta}(x).$$

Gluing maps may be used to define smooth manifolds which are not necessarily subsets of $\mathbb{R}^N$ (though we won't do so here). It is instructive to think of k-dimensional smooth manifolds as spaces that are smoothly glued together from open sets in $\mathbb{R}^k$.

Example 11. *Recall the example of the atlas of $\mathbb{S}^2$ above. Compute*

$$\begin{aligned}
\mathcal{O}_{12} &= \mathbb{S}^2 \setminus \{S, N\}, \\
\mathcal{U}_{12} &= \mathbb{R}^2 \setminus \{0\}, \\
\mathcal{U}_{21} &= \mathbb{R}^2 \setminus \{0\}, \\
z = \phi_{12}(y) &= \phi_2^{-1}(\phi_1(y)) = \left(\frac{y^1}{|y|^2}, \frac{y^2}{|y|^2} \right) = \frac{y}{|y|^2}.
\end{aligned}$$

This gluing map is called inversion *across the circle $|y|^2 = 1$ in $\mathbb{R}^2$. Each point is mapped to a point on the same ray through the origin, but the distance to the origin is replaced by its reciprocal. So we can think of $\mathbb{S}^2$ as two copies of $\mathbb{R}^2$ glued together along $\mathbb{R}^2 \setminus \{0\}$ by the inversion map across the unit circle.*

3.2 Tangent vectors on manifolds

Recall that for a solution ϕ to an autonomous system $\dot{x} = v(x)$, the parametric curve $\phi(t)$ has tangent vector $\dot{\phi}(t) = v(\phi(t))$ at time t. We will use this to define tangent vectors to manifolds. A *tangent vector* at a point p in a smooth manifold X is given by the derivative $\dot{\alpha}(0)$ of a smooth curve $\alpha\colon (-\epsilon, \epsilon) \to X \subset \mathbb{R}^N$ so that $\alpha(0) = p$. (Note the fact $\mathbb{R}^N$ is a vector space allows us to differentiate α.) The space of all tangent vectors at p is called the *tangent space* T_pX of X at p, and it is characterized by the following proposition.

Proposition 40. *If $X \subset \mathbb{R}^N$ is a k-dimensional smooth manifold, then the tangent space T_pX is the following: Given a local parametrization of X*

$$\phi\colon \mathcal{U} \to \mathcal{O} \ni p$$

so that $\phi(0) = p$,

$$T_pX = D\phi(0)(\mathbb{R}^k).$$

In particular, T_pX is naturally a k-dimensional vector space.

Proof. First of all, given a curve $\alpha\colon (-\epsilon, \epsilon) \to X$ so that $\alpha(0) = p$, we can ensure (by shrinking ϵ if necessary), that the image of α is contained in the coordinate neighborhood $\mathcal{O}$. Now

$$\alpha = \phi \circ (\phi^{-1} \circ \alpha)$$

and the chain rule shows that

$$\alpha'(0) = D\phi(0)[(\phi^{-1} \circ \alpha)'(0)] \in D\phi(0)(\mathbb{R}^k).$$

Thus we've shown $T_pX \subset D\phi(0)(\mathbb{R}^k)$.

To show $D\phi(0)(\mathbb{R}^k) \subset T_pX$, for any vector $v \in \mathbb{R}^k$, consider $\alpha(t) = \phi(tv)$ for $|t|$ small enough that the image of α is contained in $\mathcal{O}$. Then

$$\alpha'(0) = D\phi(0)v$$

and so $D\phi(0)(\mathbb{R}^k) = T_pX$. ∎

Also note the following corollary of our definition of T_pX:

Corollary 41. *T_pX is independent of the coordinate neighborhood $\mathcal{O}$ of p.*

If $f\colon X \to \mathbb{R}^m$ is a smooth map from a smooth k-dimensional manifold X, and if $p \in X$, then we define

$$Df(p)\colon T_pX \to \mathbb{R}^m$$

by using a local parametrization $\phi\colon \mathcal{U} \to X$ so that $\phi(q) = p$. Then we define

$$Df(p) = D(f \circ \phi)(q) \circ (D\phi(q))^{-1}.$$

The following exercise verifies this definition makes sense (see Guillemin and Pollack).

Homework Problem 27.

(a) Show that $D\phi(q)$ is invertible as a linear map from $\mathbb{R}^k$ to T_pX.

(b) Show that the definition of $Df(p)$ is independent of the coordinate parametrization ϕ.

(c) Show that if $f\colon X \to Y$ for $Y \subset \mathbb{R}^m$ a manifold, then $Df(p)(T_pX) \subset T_{f(p)}Y$.

Tangent vectors naturally differentiate functions at a point. So if $f: X \to \mathbb{R}$, then and the tangent vector $v = \alpha'(0)$ for a curve α so that $\alpha(0) = p$, then we may define

$$(vf)(p) = (f \circ \alpha)'(0) = Df(p)\alpha'(0) = Df(p)v.$$

This definition depends only on v, and not on the curve α used. (For each v there are many α, since v only depends on the first derivative $\alpha'(0)$ and no higher Taylor coefficients.)

For a coordinate system

$$\phi^{-1} = (x^1, \dots, x^k)\colon \mathcal{O} \to \mathbb{R}^k,$$

(where we assume as usual that $\phi(0) = p$), then the *coordinate basis* of T_pX induced by ϕ may be written as $\{\partial/\partial x^i\}$, which are thought of as tangent vectors differentiating functions f by

$$\left(\frac{\partial}{\partial x^i}\right) f\bigg|_p = \frac{\partial}{\partial x^i}\, f \circ \phi^{-1}\bigg|_0 = \frac{\partial}{\partial x^i} f(x^1, \dots, x^k)\bigg|_0.$$

($\partial/\partial x^i$ is the tangent vector associated to the curve $\alpha = \phi(te_i)$, for e_i the i^{th} basis standard basis vector in $\mathbb{R}^k$.) Thus we can write any tangent vector v at p as

$$v = v^i \frac{\partial}{\partial x^i}.$$

Writing tangent vectors in terms of the coordinate basis of T_pX is much more useful than writing them in terms of a basis of $\mathbb{R}^N \supset T_pX$.

The components v^i will change depending on the local coordinates. On $\mathcal{O}_{\alpha\beta} = \mathcal{O}_\alpha \cap \mathcal{O}_\beta$ the intersection of two coordinate neighborhoods of p, then we have two coordinate systems $\phi_\alpha^{-1} = (x^1, \ldots, x^k)$ and $\phi_\beta^{-1} = (y^1, \ldots, y^k)$. We can write by using the chain rule

$$v = v^i(x)\frac{\partial}{\partial x^i} = v^i(x)\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} = v^j(y)\frac{\partial}{\partial y^j}.$$

Therefore, we know how the v^i change under coordinate transformations $x \to y$:

$$v^j(y) = v^i(x)\frac{\partial y^j}{\partial x^i}. \tag{22}$$

(In a more coordinate-free notation, the Jacobian matrix $\partial y^j/\partial x^i$ is the derivative of the gluing map $\phi_{\alpha\beta} = \phi_\beta^{-1} \circ \phi_\alpha$. It is easy to check that $y = \phi_{\alpha\beta} \circ x$.)

All the tangent spaces of a manifold X patch together to make a larger manifold TX called the *tangent bundle*. We define the tangent bundle

$$TX = \{(p, w) \in \mathbb{R}^N \times \mathbb{R}^N : p \in X,\, w \in T_pX\}.$$

Homework Problem 28. *If X is a k-dimensional manifold, show that TX is a $2k$-dimensional submanifold of $\mathbb{R}^{2N}$. To prove this, consider a local parametrization $\phi\colon \mathcal{U} \to X \subset \mathbb{R}^N$.*

(a) Define $\Phi\colon \mathcal{U} \times \mathbb{R}^k \to \mathbb{R}^{2N}$ for $y = (y^1, \ldots, y^k)$ by

$$\Phi(x, y) = \left(\phi(x), \frac{\partial \phi}{\partial x^i}(x)\, y^i\right).$$

Show that $\Phi(\mathcal{U} \times \mathbb{R}^k)$ is an open subset of TX and that Φ is one-to-one.

(b) Show that $D\Phi$ has rank $2k$.

(c) Show that Φ^{-1} is continuous from $\Phi(\mathcal{U} \times \mathbb{R}^k)$ to $\mathcal{U} \times \mathbb{R}^k$.

There is a natural smooth map

$$\pi\colon TX \to X, \qquad \pi(p, w) = p,$$

and each $\pi^{-1}(\{p\})$ is the vector space T_pX.

Each coordinate system $\phi^{-1} = (x^1, \dots, x^k)$, provides a *local frame* $\{\partial/\partial x^i\}$ of the tangent bundle. A local frame is a basis of the tangent space for every p in a neighborhood $\mathcal{O} \subset X$. These frames are patched together in the following paragraph.

A more abstract view of the tangent bundle is given by looking a given smooth atlas $\{\mathcal{O}_\alpha, \phi_\alpha, \mathcal{U}_\alpha\}$ of X. Then as a set, we may identify

$$TX = \left(\bigsqcup_\alpha \mathcal{U}_\alpha \times \mathbb{R}^k \right) / \approx,$$

where the equivalence class $\approx$ is given by

$$(x, v) \approx (y, w) \quad \text{if } x \in \mathcal{U}_{\alpha\beta},\ y \in \mathcal{U}_{\beta\alpha},\ y = \phi_{\alpha\beta}(x),\ w = D\phi_{\alpha\beta} v.$$

A *vector field* on a manifold X provides a tangent vector at every point in X. More precisely, a vector field is a *section* of the tangent bundle. In other words, $v\colon X \to TX$ is a vector field if $\pi(v(p)) = p$ for all $p \in X$. So $v(p) = (p, w(p))$ for $w(p) \in T_pX$. In fact, for $X \subset \mathbb{R}^N$, $w\colon X \to \mathbb{R}^N$ so that $w(p) \in T_pX$ is equivalent to $v(p) = (p, w(p))$. (Clearly v and w carry the same amount of information, and we often will refer to both of them using the same symbol v.)

A vector field v is smooth if it is given as a smooth map from X to $\mathbb{R}^N \times \mathbb{R}^N \supset TX$ as above. Equivalently, v is smooth if for every local coordinate system $(x^1, \dots, x^k)$,

$$v = v^i(x) \frac{\partial}{\partial x^i}$$

for v^i smooth on $\mathcal{U} \subset \mathbb{R}^k$.

3.3 Flows on manifolds

A smooth vector field v on a manifold X defines a system of ODEs in the local coordinates of X (or we may say more simply a system on X). The ODE system is given by

$$\dot{x} = v(x)$$

for $x\colon I \to X$ a parametric curve.

In order to describe the relationship between the local and global pictures of the ODE system, consider $X \subset \mathbb{R}^N$ and $v : X \to \mathbb{R}^N$ so that for each $p \in X$, $v(p) \in T_pX$. Consider a local parametrization $\phi_\alpha : \mathcal{U}_\alpha \to \mathcal{O}_\alpha$. Let $\phi_\alpha^{-1} = (x_\alpha^1, \dots, x_\alpha^k)$. Locally on $U_\alpha \subset \mathbb{R}^k$, we represent v by

$$v_\alpha = v_\alpha^i \frac{\partial}{\partial x_\alpha^i}.$$

In other words, for $p \in \mathcal{O}_\alpha \subset X$, we have

$$v(p) = D\phi_\alpha(p) v_\alpha(p).$$

Proposition 42. *Consider v a smooth vector field on $X \subset \mathbb{R}^N$. Consider a solution ψ_α to $\dot{x}_\alpha = v_\alpha(x_\alpha)$, where $\psi_\alpha \colon I \to U_\alpha$ for a time interval I. Then*

$$\psi = \phi_\alpha \circ \psi_\alpha$$

is a solution to $\dot{x} = v(x)$ from I to $\mathcal{O}_\alpha \subset X$. Every solution to $\dot{x} = v(x)$ restricted to $\mathcal{O}_\alpha$ is of this form.

Proof. First of all, note that $\dot{x}_\alpha = v_\alpha(x_\alpha)$ is a well-defined system of ODEs on the open set $\mathcal{U}_\alpha \subset \mathbb{R}^k$. On the other hand, on X, the system $\dot{x} = v(x)$ is not an ODE system on $\mathbb{R}^N \supset X$. This may be remedied locally as follows: For each $p \in X$, $v(p) \in T_pX \subset \mathbb{R}^N$. Then since v is a smooth function, we may locally extend v to a smooth function to $\mathbb{R}^N$ (we refer to each local extension simply as v).

Consider a solution ψ_α to $\dot{x}_\alpha = v_\alpha(x_\alpha)$. Then if we let $\psi = \phi_\alpha \circ \psi_\alpha$, then compute

$$\dot{\psi} = D\phi_\alpha(\dot{\psi}_\alpha) = D\phi_\alpha(v_\alpha) = v.$$

Thus ψ is a solution. To show that every solution ψ to $\dot{x} = v(x)$ is of this form, note that since T_pX is the image of $D\phi_\alpha(q)$ for $\phi_\alpha(q) = p$ (Proposition 40), then every smooth vector field v is locally equal to $D\phi_\alpha v_\alpha$. Then by uniqueness of ODEs, the solution to $\dot{x} = v(x)$ must be the image of the solution to $\dot{x}_\alpha = v_\alpha(x_\alpha)$. □

Remark. The restriction to autonomous equations $\dot{x} = v(x)$ is unnecessary. The same proof works for non-autonomous systems $\dot{x} = v(x, t)$ on manifolds.

Recall a subset X of a metric space Y is *compactly contained* in another subset Z if $\bar{X}$ is compact and $\bar{X} \subset Z$. In this case we write $X \subset\subset Z$, and say X is a *precompact* subset of Z.

Theorem 15. *Let v be a smooth vector field on a compact manifold X. Then the flow $F(y,t)$ along the vector field (the solution to*

$$\dot{x} = v(x), \qquad x(0) = y)$$

is a smooth function from $X \times \mathbb{R}$ to X. In particular, any flow on a compact manifold exists for all time.

Proof. Consider an atlas $\{\mathcal{O}_\alpha, \phi_\alpha, \mathcal{U}_\alpha\}$ of X. First of all, by Lemma 43 below, there is an open cover $\mathcal{Q}_\beta$ of X so that each $\mathcal{Q}_\beta \subset\subset \mathcal{O}_\alpha$ for some $\mathcal{O}_\alpha$ in the atlas. Then each $\phi_\alpha^{-1}\mathcal{Q}_\beta$ is a compact subset of $\mathcal{U}_\alpha$. Our differential equation is equivalent to $\dot{x}_\alpha = v_\alpha(x_\alpha)$ on each $\mathcal{U}_\alpha$.

Since X is compact, we can choose a finite subcover $\{\mathcal{Q}_1, \dots, \mathcal{Q}_n\}$ of the open cover $\{\mathcal{Q}_\beta\}$. For each $i = 1, \dots, n$, an straightforward analog of Lemma 23 shows there is an $\epsilon_i > 0$ so that if $x_0 \in \phi_\alpha^{-1}\mathcal{Q}_i$, then the solution to

$$\dot{x}_\alpha = v_\alpha(x_\alpha), \qquad x(0) = x_0$$

stays in $\mathcal{U}_\alpha$ for $t \in [-\epsilon_i, \epsilon_i]$. Moreover, by Proposition 31, for any $T \in \mathbb{R}$, the solution with initial condition $x(T) = x_0 \in \phi_\alpha^{-1}\mathcal{Q}_i$ stays within $\mathcal{U}_\alpha$ for time $t \in [T - \epsilon_i, T + \epsilon_i]$.

Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\} > 0$. Then for every $T \in \mathbb{R}$, $p \in X$, we claim the solution to $\dot{x} = v(x)$, $x(T) = p$ exists for all $t \in [T - \epsilon, T + \epsilon]$. To prove the claim, note that each $p \in X$ lies in one of the $\mathcal{Q}_i \subset \mathcal{O}_\alpha$, and that the solution to

$$\dot{x}_\alpha = v_\alpha(x_\alpha), \qquad x(T) = \phi_\alpha^{-1}(p)$$

lies in $\mathcal{U}_\alpha$ for $t \in [T - \epsilon, T + \epsilon]$. Thus Proposition 42 shows that the solution to $\dot{x} = v(x)$, $x(T) = p$ is in $\mathcal{O}_\alpha$ for $t \in [T - \epsilon, T + \epsilon]$, and the claim is proved.

In order to prove the Theorem, continue as in the proof of Lemma 25 to show the solution exists for all time. The smoothness of the solution follows from Theorem 12 and Proposition 42. □

Lemma 43. *Given an atlas $\{\mathcal{O}_\alpha, \phi_\alpha, \mathcal{U}_\alpha\}$ of a manifold X, there is an open cover $\{\mathcal{Q}_\beta\}$ of X so that each $\mathcal{Q}_\beta$ is precompact in some $\mathcal{O}_\alpha$.*

Proof. We can cover each open $\mathcal{U}_\alpha \subset \mathbb{R}^k$ by open balls $B_\beta \subset\subset \mathcal{U}_\alpha$. Then $\mathcal{Q}_\beta = \phi_\alpha(B_\beta)$ forms an open cover of X. □

The *support* of an $\mathbb{R}^m$-valued function f is the closure

$$\mathrm{supp}(f) = \overline{\{x : f(x) \neq 0\}}.$$

An important class of functions is smooth functions with compact support. Prominent examples can be constructed using the smooth function on $\mathbb{R}$

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

See the notes on bump functions.

Homework Problem 29. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Consider a smooth vector field $v \colon \Omega \to \mathbb{R}^n$ with compact support. Show that any solution ψ to $\dot{x} = v(x)$, $x(0) = x_0 \in \Omega$, exists for all time $t \in \mathbb{R}$.*

Hint: First show that if $v(y) = 0$, then any solution to $\dot{x} = v(x)$, $x(t_0) = y$, must be constant for all time. Use this to show that any solution to $\dot{x} = v(x)$, $v(x(0)) \neq 0$, must remain in $\mathrm{supp}(v)$ *for its entire maximal interval of definition. Apply Theorem 7.*

Given a smooth manifold X, consider the set $\mathrm{Diff}(X)$ of diffeomorphisms from X to itself. Then for $f, g \in \mathrm{Diff}(X)$, it is easy to see that

$$f \circ g \in \mathrm{Diff}(X), \quad f^{-1} \in \mathrm{Diff}(X), \quad f \circ f^{-1} = \mathrm{id}$$

for id the identity map. Therefore, $\mathrm{Diff}(X)$ is a group.

Proposition 44. *Let v be a smooth vector field on a compact manifold X. Then for the flow $F(y,t)$, define $F_t(y) = F(y,t)$. Then $F_t \in \mathrm{Diff}(X)$, $F_{t_1+t_2} = F_{t_1} \circ F_{t_2}$, and $F_{-t} = F_t^{-1}$. (And so F is a* group homomorphism *from the additive group $\mathbb{R}$ to $\mathrm{Diff}(X)$.)*

Proof. Theorem 15 shows that F_t is smooth for any t. The group homomorphism property is simply a restatement of Proposition 34. Therefore, $F_t \circ F_{-t} = F_0$, which is the flow along v for time 0. By definition, $F_0 = \mathrm{id}$ the identity map. Now $F_t^{-1} = F_{-t}$ is smooth, and so F_t is a diffeomorphism. □

Remark. Note the only place we used the fact that X is compact is to guarantee the existence of the flow for all time. So the proposition still holds for any smooth vector field v on a smooth manifold X so that the flow exists for all time.

Example 12. *For the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, consider the vector field defined by $v(x^1, x^2, x^3) = (-x^2, x^1, 0)$. It is straightforward to show that the tangent space to $\mathbb{S}^2$ at (x^1, x^2, x^3) is given by $v = (v^1, v^2, v^3) \in \mathbb{R}^3$ so that $v^1x^1 +$*

$v^2x^2 + v^3x^3 = 0$. (Proof: $\mathbb{S}^2 = \{f = 1\}$ for $f = (x^1)^2 + (x^2)^2 + (x^3)^2$, and so for any local parametrization ϕ, we have $f \circ \phi = 1$. Thus the Chain Rule shows that $Df(x)(T_x\mathbb{S}^2) = 0$, and so $T_x\mathbb{S}^2 \subset \ker Df(x)$. They must be equal since both are two-dimensional vector spaces. Then simply compute $\ker Df(x)$.) Therefore, v is a smooth vector field on $\mathbb{S}^2$.

Recall that the coordinate systems of the atlas introduced above are

$$\begin{aligned}(y^1, y^2) = \phi_1^{-1}(x^1, x^2, x^3) &= \left(\frac{x^1}{1-x^3}, \frac{x^2}{1-x^3}\right),\\ (z^1, z^2) = \phi_2^{-1}(x^1, x^2, x^3) &= \left(\frac{x^1}{1+x^3}, \frac{x^2}{1+x^3}\right).\end{aligned}$$

On $\mathcal{U}_1$, compute at $x = (x^1, x^2, x^3) \in \mathcal{O}_1 \subset \mathbb{S}^2$,

$$\begin{aligned}D\phi_1^{-1}(x)(v) &= \begin{pmatrix} \frac{1}{1-x^3} & 0 & \frac{x^1}{(1-x^3)^2} \\ 0 & \frac{1}{1-x^3} & \frac{x^2}{(1-x^3)^2} \end{pmatrix} \begin{pmatrix} -x^2 \\ x^1 \\ 0 \end{pmatrix}\\ &= \begin{pmatrix} -\frac{x^2}{1-x^3} \\ \frac{x^1}{1-x^3} \end{pmatrix} = \begin{pmatrix} -y^2 \\ y^1 \end{pmatrix}.\end{aligned}$$

It turns out that for $x \in \mathcal{O}_2$,

$$D\phi_2^{-1}(x)(v) = \begin{pmatrix} -z^2 \\ z^1 \end{pmatrix}$$

as well.

In the coordinate charts, these systems can be solved explicitly. For $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, compute the fundamental solution

$$\begin{aligned}e^{At} &= Pe^{tD}P^{-1}\\ &= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \exp\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} t\right] \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{pmatrix}\\ &= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \cos t + i\sin t & 0 \\ 0 & \cos t - i \sin t \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{pmatrix}\\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.\end{aligned}$$

Therefore, for $y \in \mathcal{U}_1$, the solution to $\dot{y} = v(y)$, $y(0) = y_0$ is

$$y(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} y_0. \tag{23}$$

And also, for $z \in \mathcal{U}_2$, the solution to $\dot{z} = v(z)$, $z(0) = z_0$ is

$$z(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} z_0. \tag{24}$$

Proposition 42 implies that these two flows should be related, since they both correspond to flows on $\mathbb{S}^2$. In particular, for $y_0 \in \mathcal{U}_{12}$, let $z_0 = \phi_{12}(y_0) = y_0|y_0|^{-2}$. Then we check that the solution

$$z(t) = \phi_{12}(y(t))$$

for $y(t)$ from (23) and $z(t)$ from (24). So compute

$$\begin{aligned}
y(t) &= \begin{pmatrix} y_0^1 \cos t - y_0^2 \sin t \\ y_0^1 \sin t + y_0^2 \cos t \end{pmatrix} \qquad \textit{for } y_0 = \begin{pmatrix} y_0^1 \\ y_0^2 \end{pmatrix}, \\
|y(t)|^2 &= (y_0^1 \cos t - y_0^2 \sin t)^2 + (y_0^1 \sin t + y_0^2 \cos t)^2 = |y_0|^2, \\
\phi_{12}(y(t)) &= \frac{y(t)}{|y(t)|^2} = \frac{1}{|y_0|^2} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} y_0 \\
&= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} z_0 = z(t).
\end{aligned}$$

Therefore, the flow patches from $\mathcal{U}_1$ to $\mathcal{U}_2$.

The flow itself can be represented by on $\mathcal{U}_1$ by

$$F_t(y) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} y,$$

on $\mathcal{U}_2$ by

$$F_t(z) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} z,$$

and even on $\mathbb{S}^2 \subset \mathbb{R}^3$ itself by

$$F_t(x) = \exp\left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} t \right] x = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} x.$$

Homework Problem 30. *Consider the atlas given above for $\mathbb{S}^2$. On $\mathcal{U}_1$, consider the vector field*

$$v = -y^1 \frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial y^2}.$$

Show that $D\phi_1 v$ is extends to a smooth vector field on all of $\mathbb{S}^2$ (i.e., it extends smoothly across $N = \mathbb{S}^2 \setminus \mathcal{O}_1$.) Write down this vector field in the z coordinates on $\mathcal{U}_2$ as well. Solve for the flow on $\mathcal{U}_1$ and $\mathcal{U}_2$, and explicitly check they agree on the overlap $\mathcal{O}_{12}$.

3.4 Riemannian metrics

For a vector v at a point p on a manifold $X \subset \mathbb{R}^N$, we can measure the length of v by using the inner product on $\mathbb{R}^N$. So if $v \in T_pX \subset \mathbb{R}^N$, and

$$v = v^a \frac{\partial}{\partial y^a}$$

for $y = (y^1, \cdots, y^N)$ coordinates on $\mathbb{R}^N$, then the length $|v|$ of v is given by

$$|v|^2 = \sum_{a=1}^{N} (v^a)^2 = \delta_{ab} v^a v^b$$

for the Kronecker $\delta_{ab} = 1$ if $a = b$ and $\delta_{ab} = 0$ if $a \neq b$. In this usage for computing the length of a tangent vector on $\mathbb{R}^N$, the Kronecker δ is a *Riemannian metric.*

(Note we use the following convention for an n-dimensional manifold $X \subset \mathbb{R}^N$: use indices a, b, c from 1 to N to represent coordinates in $\mathbb{R}^N$, and use i, j, k from 1 to n to represent local coordinates on X.)

On a manifold X, a Riemannian metric is a smoothly varying positive definite inner product on T_pX for all $p \in X$. Recall the definitions involved. An *inner product* on a real vector space V is a pairing $g\colon V \times V \to \mathbb{R}$ which is bilinear and symmetric. g is *bilinear* if for every $v \in V$, the maps $g(v, \cdot)$ and $g(\cdot, v)$ from V to $\mathbb{R}$ are linear maps, and g is *symmetric* if for each $v, w \in V$, $g(v, w) = g(w, v)$. An inner product is *positive definite* if $g(v, v) \geq 0$ for all $v \in V$ and $g(v, v) = 0$ only if $v = 0$.

If the vector space V has a basis e_i, then the inner product g is determined by $g_{ij} = g(e_i, e_j)$, since for any linear combination $v = v^i e_i$, $w = w^j e_j$, bilinearity shows

$$g(v,w) = g(v^i e_i, w^j e_j) = v^i g(e_i, w^j e_j) = v^i w^j g(e_i, e_j) = v^i w^j g_{ij}.$$

The fact g is symmetric is equivalent to $g_{ij} = g_{ji}$.

Note that a positive definite inner product g provides a way to measure the length of a vector $|v|_g = \sqrt{g(v,v)}$, and it also provides a measurement of the angle θ between two nonzero vectors v and w:

$$\cos\theta = \frac{g(v,w)}{|v|_g |w|_g}.$$

A Riemannian metric on X gives a positive definite inner product on each tangent space T_pX. We also require these inner products to vary smoothly as the point p varies in X. To describe this, consider a smooth atlas on X, and a local coordinate system $(x^1, \dots, x^k)$ around p. Then a smooth vector field v can be represented as $v = v^i \frac{\partial}{\partial x^i}$ for the standard local frame $\{\partial/\partial x^i\}$ of the tangent bundle. Then at each point, the inner product g is represented by $g_{ij}(x)$, and

$$g(v,w) = g_{ij} v^i w^j, \qquad v^i = v^i(x),\ w^j = w^j(x),\ g_{ij} = g_{ij}(x).$$

Then g is *smoothly varying* on X if the functions g_{ij} are smoothly varying on each coordinate chart in the smooth atlas of X.

Euclidean space $\mathbb{R}^N$ has a standard Riemannian metric given by the standard inner product δ_{ab}. As we've seen above, for any submanifold $X \subset \mathbb{R}^N$ endows X with a Riemannian metric. In particular, for $v, w \in T_pX \subset \mathbb{R}^N$, we can form $g(v,w)$ using the inner product δ_{ab}. In particular, consider a smooth parametrization $\phi \colon \mathcal{U} \to \mathcal{O} \subset X \subset \mathbb{R}^N$. Then $\phi = (\phi^1, \cdots, \phi^N)$. A vector field represented by

$$v = v^i \frac{\partial}{\partial x^i}$$

on $\mathcal{U} \subset \mathbb{R}^n$ is represented by

$$D\phi(x)(v) = \frac{\partial \phi^a}{\partial x^i}(x) v^i(x) \in T_{\phi(x)}X \subset \mathbb{R}^N.$$

$D\phi(x)(v)$ is called the *push-forward* of v under the map ϕ. For $v, w \in T_pX$, we may define the metric

$$\begin{aligned} g_{ij} v^i w^j = g(v,w) &= \left(\frac{\partial \phi^a}{\partial x^i} v^i\right)\left(\frac{\partial \phi^b}{\partial x^j} w^j\right)\delta_{ab} \\ &= \left(\frac{\partial \phi^a}{\partial x^i}\frac{\partial \phi^b}{\partial x^j}\delta_{ab}\right) v^i w^j. \end{aligned}$$

Therefore, the Euclidean inner product on $\mathbb{R}^N$ induced the Riemannian metric on X locally given by the formula

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij} = \frac{\partial \phi^a}{\partial x^i}\frac{\partial \phi^b}{\partial x^j}\delta_{ab}. \tag{25}$$

Given a real vector space V, the *dual vector space* V^* is given by the set of all linear functions from V to $\mathbb{R}$. It is easy to check V^* is a vector space. If V has a basis $\{e_i\}$, then there is a dual basis $\{\eta^i\}$ of V^*, which is defined as follows:

$$\eta^i(e_j) = \delta^i_j.$$

Given a local coordinate frame $\{\partial/\partial x^i\}$ of TX, the local frame on the dual space is written as $\{dx^i\}$. Each dx^i is called a *differential*. The dual space T_p^*X of T_pX is called the *cotangent space* of X at p.

Lemma 45. *If $y = y(x)$ is a coordinate change as in (22), then*

$$dy^j = \frac{\partial y^j}{\partial x^i}dx^i.$$

Proof. Write $dy^j = \xi^j_\ell dx^\ell$. Then we have

$$\delta^j_i = dy^j\left(\frac{\partial}{\partial y^i}\right) = \xi^j_\ell dx^\ell\left(\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^k}\right) = \xi^j_\ell\frac{\partial x^k}{\partial y^i}dx^\ell\left(\frac{\partial}{\partial x^k}\right) = \xi^j_\ell\frac{\partial x^k}{\partial y^i}\delta^\ell_k = \xi^j_k\frac{\partial x^k}{\partial y^i}.$$

Therefore, (ξ^j_k) is the inverse matrix of $\left(\frac{\partial x^k}{\partial y^i}\right)$, and so $\xi^j_k = \frac{\partial y^j}{\partial x^k}$. □

A Riemannian metric can be naturally written as

$$g_{k\ell}\, dy^k dy^\ell = g_{ij}\frac{\partial y^k}{\partial x^i}\frac{\partial y^\ell}{\partial x^j}\, dx^i dx^j.$$

This makes sense because of the natural pairing

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$$

between the tangent and cotangent spaces implies that

$$g(v,w) = g_{ij}\, dx^i\left(v^k\frac{\partial}{\partial x^k}\right) dx^j\left(w^\ell\frac{\partial}{\partial x^\ell}\right) = g_{ij}(v^k\delta^i_k)(w^\ell\delta^j_\ell) = g_{k\ell}\, v^k w^\ell.$$

A Riemannian metric is an example of a *tensor* on X. The *tensor product* $V \otimes W$ of two real vector spaces with bases respectively ν_i and ω_j is the real vector space formed from the basis

$$\{\nu_i \otimes \omega_j\}.$$

This implies

$$\dim V \otimes W = (\dim V)(\dim W).$$

A tensor of type (k, ℓ) on a manifold X assigns to each point $p \in X$ an element of

$$(T_p X)^{\otimes k} \otimes (T_p^* X)^{\otimes \ell},$$

which has as its basis

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell} \right\}.$$

Locally, we write a tensor ω as

$$\omega^{i_1 \cdots i_k}_{j_1 \cdots j_\ell} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell},$$

or simply as $\omega^{i_1 \cdots i_k}_{j_1 \cdots j_\ell}$. We say ω is smooth if each $\omega^{i_1 \cdots i_k}_{j_1 \cdots j_\ell}$ is smooth locally for all coordinates in a smooth atlas of X.

A Riemannian metric is then a smooth symmetric $(0, 2)$ tensor on a manifold X. Since the product is symmetric, we omit the $\otimes$ and simply write $g_{ij}\, dx^i dx^j$ for a Riemannian metric in local coordinates x. (There are also antisymmetric $(0, k)$ tensors, or k-forms, for which the tensor product $\otimes$ is replaced by $\wedge$.)

Example 13. *For $\mathbb{S}^2$, in the local coordinate given by stereographic projection, recall the coordinate chart $\phi = \phi_1$:*

$$\phi(y^1, y^2) = \left(\frac{2y^1}{|y|^2 + 1}, \frac{2y^2}{|y|^2 + 1}, \frac{|y|^2 - 1}{|y|^2 + 1} \right),$$

and the Riemannian metric induced from $\mathbb{R}^3$ is

$$
\begin{aligned}
g_{ij}\, dy^i dy^j &= \delta_{ab}\, \frac{\partial \phi^a}{\partial y^i} \frac{\partial \phi^b}{\partial y^j} dy^i dy^j \\
&= \delta_{ab}\, d\phi^a d\phi^b \\
&= d\phi^1 d\phi^1 + d\phi^2 d\phi^2 + d\phi^3 d\phi^3 \\
&= \left(\frac{-2(y^1)^2 + 2(y^2)^2 + 2}{(|y|^2+1)^2}\, dy^1 + \frac{-4y^1 y^2}{(|y|^2+1)^2}\, dy^2 \right)^2 \\
&\quad + \left(\frac{-4y^1 y^2}{(|y|^2+1)^2}\, dy^1 + \frac{2(y^1)^2 - 2(y^2)^2 + 2}{(|y|^2+1)^2}\, dy^2 \right)^2 \\
&\quad + \left(\frac{4y^1}{(|y|^2+1)^2}\, dy^1 + \frac{4y^2}{(|y|^2+1)^2}\, dy^2 \right)^2 \\
&= \frac{4}{(|y|^2+1)^2} (dy^1 dy^1 + dy^2 dy^2).
\end{aligned}
$$

Note in the previous example, we used the formula for differentials

$$d\phi^a = \frac{\partial \phi^a}{\partial y^i}\, dy^i.$$

It is also useful to have the following notation: If $h = h_{ab}\, dz^a dz^b$ is a Riemannian metric on Z, and $\phi\colon Y \to Z$ is a smooth map, then we denote the *pullback* metric

$$\phi^* h = h_{ab}(\phi)\, d\phi^a d\phi^b$$

on Y. Thus in the construction above, if $\delta = \delta_{ab}\, dx^a dx^b$ is the Euclidean metric on $\mathbb{R}^N$, then the metric g induced on a submanifold $\phi\colon X \hookrightarrow \mathbb{R}^N$ is the pullback $\phi^*\delta$.

Homework Problem 31. *Let $\phi\colon X \to Y$ be a smooth map of manifolds. Let Y have a Riemannian metric h on it. Show that $\phi^* h$ is a Riemannian metric on X if and only if the tangent map $D\phi(x)\colon T_x X \to T_{\phi(x)} Y$ is injective for every $x \in X$. (In this case ϕ is called an* immersion.*)*

Hint: Do the calculations in local coordinates on X and Y. The key point to check is whether $\phi^* h$ is positive definite. Show $\phi^* h(x)$ is 0 on the kernel of $D\phi(x)$.

Note in the previous example, we considered the Riemannian metric on $\mathbb{S}^2$ pulled back from the Euclidean metric on $\mathbb{R}^3$. It is possible to write down other Riemannian metrics as well.

Example 14. *Consider hyperbolic space*

$$\mathbb{H}^n = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^n > 0\}$$

equipped with the Riemannian metric

$$\frac{dx^1 dx^1 + \cdots + dx^n dx^n}{(x^n)^2}.$$

A famous theorem of John Nash shows that for every Riemannian metric g on a smooth manifold X, there is an embedding $i\colon X \to \mathbb{R}^N$ so that g is induced from the standard metric on $\mathbb{R}^N$. (Although it is not in most cases obvious what the embedding is.)

3.5 Vector bundles and tensors

In order to explain better what tensors are, we introduce the idea of a *vector bundle*. The tangent bundle TX of a smooth n-dimensional manifold X is a vector bundle. Recall there is a map

$$\pi\colon TX \to X.$$

The *fiber* over a point $p \in X$ $\pi^{-1}(p) = T_pX$ is an n-dimensional vector space. Moreover, over each coordinate neighborhood $\mathcal{O} \subset X$ with coordinates $\{x^1, \dots, x^n\}$, $\pi^{-1}\mathcal{O}$ is diffeomorphic to $\mathcal{O} \times \mathbb{R}^n$, the diffeomorphism being

$$(p, v) \mapsto (p, v^1, \dots, v^n)$$

for $p \in \mathcal{O}$, $v = v^i \frac{\partial}{\partial x^i} \in T_pX$.

We generalize these properties of TX to define a vector bundle. A vector bundle of rank k over a manifold X is given by an $n + k$ dimensional manifold V with a smooth map $\pi : V \to X$. V is called the *total space* of the vector bundle. Every point in X has a neighborhood $\mathcal{O}$ so that $\pi^{-1}\mathcal{O}$ is diffeomorphic to $\mathcal{O} \times \mathbb{R}^k$. Under this diffeomorphism, π is simply the natural projection from $\mathcal{O} \times \mathbb{R}^k \to \mathcal{O}$. Thus vector bundles are *locally trivial*, in that each vector bundle is locally a product of a neighborhood times $\mathbb{R}^k$. Note that each diffeomorphism

$$\pi^{-1}\mathcal{O} \to \mathcal{O} \times \mathbb{R}^k$$

provides for each $p \in \mathcal{O}$ a basis of the vector space $\pi^{-1}(p)$ by taking the preimage of the standard basis of $\mathbb{R}^k$ under the diffeomorphism. Such a smoothly varying basis is called a *local frame* of the vector bundle over $\mathcal{O}$.

Given a gluing map $y = y(x)$ of two small coordinate neighborhoods $\mathcal{O}_x$ and $\mathcal{O}_y$ in X, there is a corresponding gluing map of $\mathcal{O}_x \times \mathbb{R}^k$ and $\mathcal{O}_y \times \mathbb{R}^k$. We require this gluing map to be of the form

$$(x, v) \mapsto (y(x), A(x)v)$$

for v a vector in $\mathbb{R}^k$ and $A(x)$ a smoothly varying nonsingular matrix in x. Therefore, above each point p, if we change coordinates from x to y, the frame changes by the matrix $A(x)$. $A(x)$ is a *transition function* of the vector bundle V. So the transition functions act on the fibers of a vector bundle as linear isomorphisms. This preserves the vector-space structure on each fiber when changing coordinates.

Remark. We have defined real vector bundles of rank k, for which each fiber is diffeomorphic to $\mathbb{R}^k$. We may also define *complex vector bundles* with fibers diffeomorphic to $\mathbb{C}^k$.

A *section* of a vector bundle $\pi\colon V \to X$ is a map $s\colon X \to V$ satisfying $\pi(s(p)) = p$ for all $p \in X$. So for each $p \in X$, $s(p)$ is an element of the vector space $\pi^{-1}(p)$. A vector field is precisely a section of the tangent bundle. Locally, k sections which are linearly independent on each fiber form a frame of the vector bundle. For example, $\{\partial/\partial x^i\}$ are n linearly independent sections of the tangent bundle over a coordinate chart.

Since vector bundles preserve the linear structure on each fiber, we may do linear algebra on the fibers to create new vector bundles. In particular, we can take duals and tensor products of the fiber space to form new vector bundles. The *tensor bundle* of type (k, ℓ) over an n dimensional manifold X is the vector bundle of rank $n^{k+\ell}$ with the fiber over p given by

$$T_pX^{\otimes k} \otimes T_p^*X^{\otimes \ell}.$$

Over each coordinate chart, the natural frame of the tensor bundle is

$$\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}$$

for $i_1, \ldots, i_k, j_1, \ldots, j_\ell \in \{1, \ldots, n\}$. The transition functions of a tensor bundle are determined by the formulas

$$\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k}, \qquad dx^j = \frac{\partial x^j}{\partial y^\ell}\, dy^\ell.$$

For example the transition functions for the $(0,2)$ tensor bundle are given by

$$dx^i\,dx^j = \frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^\ell}\,dy^k\,dy^\ell.$$

Note we can view

$$\frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^\ell}$$

as a nonsingular $n^2 \times n^2$ matrix, which is the *tensor product* of the matrix $\frac{\partial x^i}{\partial y^k}$ with itself.

A smooth *tensor* of type (k,ℓ) is a smooth section of the (k,ℓ) tensor bundle. Thus a Riemannian metric is a smooth symmetric, positive-definite $(0,2)$ tensor.

3.6 Integration and densities

We begin by introducing the Change of Variables Formula for multiple integrals:

Theorem 16 (Change of Variables). *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $g\colon \Omega \to \mathbb{R}^n$ be one-to-one and locally C^1. Then for every L^1 function f on $g(\Omega)$ with Lebesgue measure dx and dy,*

$$\int_{g(\Omega)} f(y)\,dy = \int_\Omega f(g(x))|\det Dg(x)|\,dx.$$

Proof. See Spivak *Calculus on Manifolds.* □

Here is another useful concept. Given an open cover $\{\mathcal{O}_\alpha\}$ of a smooth manifold X, a *partition of unity* subordinate to the cover is a collection of smooth functions $\rho_\beta\colon X \to \mathbb{R}$ satisfying

1. $\rho_\beta(x) \in [0,1]$.
2. For each ρ_β, there is an α so that $\operatorname{supp}(\rho_\beta) \subset\subset \mathcal{O}_\alpha$.
3. Every $x \in X$ has a neighborhood which intersects only finitely many of the supports of the ρ_β.
4. $\sum_\beta \rho_\beta(x) = 1$.

Proposition 46. *For every open cover of a smooth manifold X, there exists a subordinate partition of unity.*

For a proof, see Spivak or Guillemin and Pollack.

Theorem 17. *A Riemannian metric g on a manifold X provides a measure on X called the* Riemannian density*.*

The construction of this measure follows below, along with a sketch of a proof.

Let $\{\mathcal{O}_\alpha, \phi_\alpha, \mathcal{U}_\alpha\}$ be a smooth atlas of X. A function $f : X \to \mathbb{R}$ is measurable if each $f \circ \phi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}$ is measurable. For a Riemannian metric g on X, the density dV_g is defined first for measurable functions $f : X \to \mathbb{R}$ whose supports are contained in some $\mathcal{O}_\alpha$. In this case, define

$$\int_X f\, dV_g = \int_{\mathcal{O}_\alpha} f\, dV_g = \int_{\mathcal{U}_\alpha} f(x)\sqrt{\det g_{ij}(x)}\, dx$$

for local coordinate x on $\mathcal{O}_\alpha$ and Lebesgue measure dx on $\mathcal{U}_\alpha \subset \mathbb{R}^n$.

The key point is to make sure this definition makes sense for functions f whose support is contained in two open charts $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$. As above, let x be the local coordinates on $\mathcal{O}_\alpha$, and let y be the coordinates on $\mathcal{O}_\beta$. Then we use the rule (25) for changing g_{ij} under a change $y = y(x)$ and the Change of Variables Theorem 16 to show

$$\begin{aligned}
\int_{\mathcal{U}_\beta} f(y)\sqrt{\det g_{ij}(y)}\, dy &= \int_{\mathcal{U}_\alpha} f(x)\sqrt{\det g_{ij}(y)}\, \left|\det \frac{\partial y^i}{\partial x^j}\right|\, dx \\
&= \int_{\mathcal{U}_\beta} f(x)\sqrt{\det\left(g_{k\ell}(x)\frac{\partial x^k}{\partial y_i}\frac{\partial x^\ell}{\partial y^j}\right)}\, \left|\det \frac{\partial y^i}{\partial x^j}\right|\, dx \\
&= \int_{\mathcal{U}_\beta} f(x)\sqrt{\det g_{k\ell}(x)}\, \left|\det \frac{\partial x^k}{\partial y^i}\right|\, \left|\det \frac{\partial y^i}{\partial x^j}\right|\, dx \\
&= \int_{\mathcal{U}_\beta} f(x)\sqrt{\det g_{k\ell}(x)}\, dx.
\end{aligned}$$

Let ρ_β be a partition of unity subordinate to the atlas $\mathcal{O}_\alpha$ of X. For any measurable subset $\Omega \subset X$, consider its characteristic function χ_Ω. Then

$$V_g(\Omega) = \int_X \chi_\Omega\, dV_g = \sum_\beta \int_X \rho_\beta \chi_\Omega\, dV_g.$$

The calculation in the previous paragraph can be used to ensure that this definition is independent of the atlas and partition of unity used. It is straightforward to check that dV_g defines a measure on X. Then for any L^1 function f on X (measured by dV_g of course),

$$\int_X f\,dV_g = \sum_\beta \int_X \rho_\beta f\,dV_g.$$

Homework Problem 32. *Check that V_g is a measure on X.*

Remark. To complete a proof of Theorem 17, it is necessary to check that the definition depends only on g and not on the atlas $\{\mathcal{O}_\alpha, \phi_\alpha, \mathcal{U}_\alpha\}$ or the partition of unity $\{\rho_\beta\}$ subordinate to the open cover $\{\mathcal{O}_\alpha\}$.

If Ω is a domain in $\mathbb{R}^n$ with smooth boundary, then the measure on the boundary $\partial\Omega$ is given by the restriction of the Riemannian metric on $\mathbb{R}^n$. (So this gives a Riemannian metric on $\partial\Omega$, and thus a density as above.) If $\partial\Omega$ is locally given by the graph of a function $(x^1, \ldots, x^{n-1}, f(x^1, \ldots, x^{n-1}))$, then

$$\phi(x^1, \ldots, x^{n-1}) = (x^1, \ldots, x^{n-1}, f(x^1, \ldots, x^{n-1}))$$

is a local parametrization of the $n-1$ dimensional manifold $\partial\Omega \subset \mathbb{R}^n$. The matrix

$$D\phi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ f_{,1} & f_{,2} & \cdots & f_{,n-1} \end{pmatrix}.$$

Then the pullback metric

$$\begin{aligned} \sum_{i,j=1}^{n-1} g_{ij}\,dx^i dx^j &= \phi^*\delta = \delta_{ab}\,d\phi^a d\phi^b \\ &= (dx^1)^2 + \cdots + (dx^{n-1})^2 + (f_{,1}dx^1 + \cdots + f_{,n-1}dx^{n-1})^2. \end{aligned}$$

As a matrix,

$$(g_{ij}) = (\delta_{ij} + f_{,i}f_{,j}).$$

In order to compute the volume form, we should compute $\det g_{ij}$. Fortunately, it is easy to compute in this case

$$\det g = 1 + |df|^2 = 1 + f_{,1}^2 + \cdots + f_{,n-1}^2,$$

(see Problem 33) below. So the density

$$dV_g = \sqrt{1 + |df|^2}\, dx_{n-1}$$

for dx_{n-1} Lebesgue measure on $\mathbb{R}^{n-1}$.

Homework Problem 33. *For w an n-dimensional column vector, and I the $n \times n$ identity matrix, show that* $\det(I + ww^\top) = 1 + |w|^2$.

Hint: Show that $I + ww^\top$ can be diagonalized, with one eigenvalue $1+|w|^2$, and with the eigenvalue 1 repeated $n-1$ times. (For this last step, show that on the $n-1$ space orthogonal to the natural $(1+|w|^2)$-eigenvector, $I + ww^\top$ acts as the identity. What is a natural eigenvector to try?)

For a function $f : \Omega \to \mathbb{R}$, the *differential*, or *one-form*, $df = \frac{\partial f}{\partial x^i} dx^i$. Under a change of coordinates $y = y(x)$, df transforms as via the chain rule

$$\frac{\partial f}{\partial y^j}\, dy^j = df = \frac{\partial f}{\partial x^i}\, dx^i = \frac{\partial f}{\partial y^j} \frac{\partial y^j}{\partial x^i}\, dx^i.$$

In particular, this gives the formula for differentials (cf. Lemma 45)

$$dy^j = \frac{\partial y^j}{\partial x^i}\, dx^i.$$

It also shows that for each $p \in X$ a manifold, we can think of $df(p) \in T_p^* X$ the cotangent space. This is investigated further in the following problem:

Homework Problem 34. *If f is a smooth function on X and v is a smooth vector field, show that at each point $p \in X$,*

$$(vf)(p) = df(p)(v(p)).$$

(In the expression on the right, consider $df(p)$ as an element of the dual space $T_p^* X$.)

Hint: Check it in a single coordinate chart.

On a Riemannian manifold (X, g) (i.e., g is a Riemannian metric on the manifold X), for each smooth function f, there is a vector field called the *gradient* of f. We define the gradient ∇f in local coordinates to be

$$(\nabla f)^i = g^{ij} f_{,j}, \qquad g^{k\ell} g_{\ell m} = \delta^k_m.$$

(So g^{ij} is the inverse of the matrix g_{ij}.) Note that the Einstein convention with one index up (typically) indicates that ∇f is a vector field.

Homework Problem 35. *Show that ∇f transforms as a vector field under coordinate changes. In other words, check that if $y = y(x)$,*

$$(\nabla f)^j(y) = \frac{\partial y^j}{\partial x^i}(\nabla f)^i(x)$$

as in (22).

Hint: First check how the inverse of the metric g^{ij} transforms. Note that in the definition $g^{ij}g_{jk} = \delta^i_k$, δ^i_k is independent of coordinate changes.

In the case of Euclidean space, it is common to use the gradient of a function instead of its differential. In this case, $\nabla f = \delta^{ab} f_{,a}$. Note that on any Riemannian manifold

$$|df|^2 = g^{ab} f_{,a} f_{,b} = g^{ac} g_{cd} g^{db} f_{,a} f_{,b} = g_{cd} (\nabla f)^c (\nabla f)^d = |\nabla f|^2.$$

Let $v = v^i \frac{\partial}{\partial x^i}$ be a vector field on a domain in $\mathbb{R}^n$. Then the *divergence* of v is a function defined to be

$$\nabla \cdot v = \frac{\partial v^i}{\partial x^i}.$$

The divergence of a vector field may also be defined on Riemannian manifolds, but the definition is somewhat more involved.

Here is another important theorem, which is a consequence of Stokes's Theorem (see Spivak, Guillemin and Pollack, or Taylor). We only state it for domains in $\mathbb{R}^n$, and not in its more general context of compact manifolds with boundary.

Theorem 18 (Divergence Theorem). *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$. Then for any C^1 vector field v on $\bar{\Omega}$,*

$$\int_\Omega \nabla \cdot v \, dx_n = \int_{\partial\Omega} v \cdot \mathbf{n} \, dV.$$

(Here $\mathbf{n}$ is the unit outward normal vector field to $\partial\Omega$, and dV is the measure on $\partial\Omega$ induced from the Euclidean metric.)

Remark. The way we have put the integration depends on the Euclidean metric (to form the dot product, dV and $\mathbf{n}$). In the general form of Stokes's Theorem, it it unnecessary to use the metric. (We may recast v and $\nabla \cdot v$ as differential forms.)

Idea of proof. We do the computation in a very special case, for v having compact support in Ω, which is the lower half-space $\{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^n \le 0\}$.

In this case the unit normal vector $\mathbf{n} = (0, \dots, 0, 1)$ and $dV = dx_{n-1}$ Lebesgue measure on $\mathbb{R}^{n-1} = \{x^n = 0\}$. Then, using Fubini's Theorem, we want to prove

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial v^i}{\partial x^i} \, dx^n dx^{n-1} \cdots dx^1 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} v^n \, dx^{n-1} \cdots dx^1.$$

Note that the left-hand integral is a sum from $i = 1$ to n. For $i = n$, compute

$$\begin{aligned} \int_{-\infty}^{0} \frac{\partial v^n}{\partial x^n} \, dx^n &= v^n(x^1, \dots, x^{n-1}, 0) - \lim_{t \to -\infty} v(x^1, \dots, x^{n-1}, t) \\ &= v^n(x^1, \dots, x^{n-1}, 0) \end{aligned}$$

since v has compact support. On the other hand, for $i \ne n$,

$$\int_{-\infty}^{\infty} \frac{\partial v^i}{\partial x^i} \, dx^i = 0$$

since v has compact support. Therefore, using Fubini's Theorem, for each $i \ne n$, we can integrate $\partial v^i / \partial x^i$ with respect to x^i first to get zero. The remaining term is the case $i = n$, and so

$$\begin{aligned} &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial v^i}{\partial x^i} \, dx^n dx^{n-1} \cdots dx^1 \\ &\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial v^n}{\partial x^n} \, dx^n dx^{n-1} \cdots dx^1 \\ &\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} v^n \, dx^{n-1} \cdots dx^1. \end{aligned}$$

This proves the Divergence Theorem in this special case.

The general case can be reduced to this special case by using a partition of unity and the Implicit Function Theorem (see Spivak). In particular, near each point in $\partial\Omega$, there is a local diffeomorphism of $\bar{\Omega}$ to the lower half-space, sending the boundary to the boundary. Together with open subsets of Ω, these form an open cover of the compact $\bar{\Omega}$, and so we may take a finite subcover, and a partition of unity subordinate to this subcover. Then we can apply the above special case to ρv for ρ in the partition of unity and v the vector field.

It is also necessary to make sure that the various terms in the integrals transform well with respect to the local diffeomorphisms. This can be checked directly, but it is better to use the language of differential forms (see Spivak or Guillemin and Pollack). □

Homework Problem 36. *Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary. On a neighborhood $\mathcal{N} \subset \mathbb{R}^n$ of a point in the boundary $\partial\Omega$, assume that*

$$\Omega \cap \mathcal{N} = \{x \in \mathcal{N} : x^n < f(x^1, \ldots, x^{n-1})\}$$

so that Ω is locally the region under the graph of a smooth function f. Compute $\mathbf{n}$ and dV. For a smooth vector field v, compute

$$\int_{\partial\Omega\cap\mathcal{N}} v \cdot \mathbf{n}\, dV$$

in terms of the integral of a function times Lebesgue measure on $\mathbb{R}^{n-1}$.

Hint: Locally, $\partial\Omega$ is a submanifold of $\mathbb{R}^n$ which is the image of

$$\phi(x^1, \ldots, x^{n-1}) = (x^1, \ldots, x^{n-1}, f(x^1, \ldots, x^{n-1})).$$

Show that $\mathbf{n}$ is proportional to $\nabla\psi$, for

$$\psi(x^1, \ldots, x^n) = x^n - f(x^1, \ldots, x^{n-1}).$$

Your answer should be of the form

$$\int_{\phi^{-1}(\partial\Omega\cap\mathcal{N})} h\, dx_{n-1}$$

for h a function of $x^1, \ldots, x^{n-1}$.

Corollary 47 (Integration by Parts). *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$. Then for any C^1 vector field v on $\bar{\Omega}$ and C^1 function f on $\bar{\Omega}$,*

$$\int_\Omega v \cdot \nabla f\, dx_n = -\int_\Omega f\, \nabla \cdot v\, dx_n + \int_{\partial\Omega} f\, v \cdot \mathbf{n}\, dV.$$

Proof. It is easy to check that $\nabla \cdot (fv) = (\nabla f) \cdot v + f\, \nabla \cdot v$, and

$$\int_\Omega \nabla \cdot (fv)\, dx_n = \int_{\partial\Omega} f\, v \cdot \mathbf{n}\, dV.$$

□

3.7 The ϵ-Neighborhood Theorem

Theorem 19. *Let $X \subset \mathbb{R}^n$ be a compact k-dimensional manifold. Then there is an $\epsilon > 0$ so that for*

$$X^\epsilon = X + B_\epsilon(0) = \{y \in \mathbb{R}^n : \text{ there is an } x \in X \text{ so that } |x - y| < \epsilon\},$$

there is a smooth projection map from X^ϵ to X which restricts to the identity on X.

Before we prove Theorem 19, we need to introduce the *normal bundle* NX, which is a vector bundle over X for $X \subset \mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$. Define

$$NX = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \in X, \langle y, z \rangle = 0 \text{ for all } z \in T_x X\}.$$

Then NX is a vector bundle of rank $n - k$, with $\pi : NX \to X$ given by $\pi\colon (x, y) \mapsto x$. For a given $x \in X$, $N_x X = \pi^{-1}(x)$ is the *normal space* to X at x, which consists of all vectors in $\mathbb{R}^n$ perpendicular to the tangent space $T_x X$.

First of all, we show that NX is a smooth n-dimensional manifold.

Homework Problem 37. *NX is a smooth manifold of dimension n.*

(a) Show that $X \subset \mathbb{R}^n$ is a smooth manifold if and only if for each $x \in X$, there is a neighborhood W of x in $\mathbb{R}^n$ and a smooth function $\psi\colon W \to \mathbb{R}^{n-k}$ so that $D\psi$ has constant rank $n - k$ and $X \cap W = \psi^{-1}(0)$. (To show $\Longrightarrow$, use Theorem 14, and to show $\Longleftarrow$, use the Implicit Function Theorem.)

(b) At each $x \in X$, and given a smooth function ψ as above, show that the normal space N_x is the image of of the transpose of the tangent map $D\psi(x)^\perp\colon \mathbb{R}^{n-k} \to \mathbb{R}^n$.

(c) Use the previous section and the techniques of Problem 28 to show NX is a manifold.

We will prove the ϵ-Neighborhood Theorem by showing that there is a neighborhood of X in $\mathbb{R}^n$ which is diffeomorphic to the a neighborhood of the zero section $\{(x, 0) : x \in X\} \subset NX$, and the map required by the ϵ-Neighborhood Theorem then comes from $\pi\colon NX \to X$.

Proof of the ϵ-Neighborhood Theorem. Consider the map $F : NX \to \mathbb{R}^n$ given by $F : (x, y) \mapsto x + y$. For each $x \in X$, $DF(x, 0) : T_x(NX) \to \mathbb{R}^n$ is a linear isomorphism. This can be proved since $T_{(x,0)}(NX)$ can be written as a sum $T_x(X) + N_x(X)$, and $DF(x)$, when restricted to each factor, is a linear isomorphism. The Inverse Function Theorem then shows that each $x \in X$, there are neighborhoods $\mathcal{N}_x$ of $(x, 0)$ in NX and W_x of x in $\mathbb{R}^n$ so that $F|_{\mathcal{N}_x}$ is a diffeomorphism from $\mathcal{N}_x$ to W_x. Note we may apply the Inverse Function Theorem because by considering a local parametrization of NX, and diffeomorphisms of (open subsets of) manifolds are defined in terms of these parametrizations.

Consider the following lemma:

Lemma 48. *There are open sets $\mathcal{N}$ and $\tilde{X}$ so that $X \times \{0\} \subset \mathcal{N} \subset NX$ and $X \subset \tilde{X} \subset \mathbb{R}^n$ and the restriction of F is a diffeomorphism from $\mathcal{N}$ to $\tilde{X}$.*

Proof. First of all, we note that DF is a linear isomorphism on $\mathcal{N}' = \bigcup_{x \in X} \mathcal{N}_x$. The Inverse Function Theorem then shows that $F|_{\mathcal{N}'}$ is a diffeomorphism onto its image as long as it is one-to-one. Therefore, we need only find an open $\mathcal{N}$ satisfying $X \times \{0\} \subset \mathcal{N} \subset \mathcal{N}'$ on which F is one-to-one.

Now assume by contradiction that no such $\mathcal{N}$ exists. Then there are points $(x_n, y_n) \neq (x_n', y_n') \in NX$ satisfying $F(x_n, y_n) = F(x_n', y_n')$ and so that $|y_n|, |y_n'| < \frac{1}{n}$ (Why? You must use the compactness of X.) Since X is compact, there must be a subsequence n_i so that $(x_{n_i}, y_{n_i}) \to (x, 0)$ as $i \to \infty$. Then we may take a further subsequence n_{i_j} so that $(x'_{n_{i_j}}, y'_{n_{i_j}}) \to (x', 0)$ as $j \to \infty$. For simplicity, we rename the subsequence n_{i_j} as simply n. Then the continuity of F shows that

$$x = F(x, 0) = \lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} F(x_n', y_n') = F(x', 0) = x'.$$

Since F is injective on $X \times \{0\}$, we have $x = x'$. But then $F|_{\mathcal{N}_x}$ is injective, which contradicts our assumption that $(x_n, y_n) \neq (x_n', y_n')$ for large n. Therefore, the lemma is proved. □

Now since X is compact, there is a small $\epsilon > 0$ so that $X^\epsilon \subset F(\mathcal{N})$. The projection map from $X^\epsilon \to X$ is then given by $\pi \circ F^{-1}$, which is smooth. This completes the proof of the ϵ-Neighborhood Theorem. □