AFFINE HERMITIAN-EINSTEIN METRICS

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1. Introduction

A holomorphic vector bundle $E \to N$ over a compact Kähler manifold (N, ω) is called *stable* if every coherent holomorphic subsheaf F of E satisfies

$$0 < \text{rank } F < \text{rank } E \qquad \Longrightarrow \qquad \mu_\omega(F) < \mu_\omega(E),$$

where μ_ω is the ω-*slope* of the sheaf given by

$$\mu_\omega(E) = \frac{\deg_\omega(E)}{\text{rank } E} = \frac{\int_N c_1(E,h) \wedge \omega^{n-1}}{\text{rank } E}.$$

Here $c_1(E,h)$ is the first Chern form of E with respect to a Hermitian metric h. The famous theorem of Donaldson [7, 8] (for algebraic manifolds only) and Uhlenbeck-Yau [24, 25] says that an irreducible vector bundle $E \to N$ is ω-stable if and only if it admits a Hermitian-Einstein metric (i.e. a metric whose curvature, when the 2-form part is contracted with the metric on N, is a constant times the identity endomorphism on E). This correspondence between stable bundles and Hermitian-Einstein metrics is often called the Kobayashi-Hitchin correspondence.

An important generalization of this theorem is provided by Li-Yau [15] for complex manifolds (and subsequently due to Buchdahl by a different method for surfaces [3]). The major insight for this extension is the fact that the degree is well-defined as long as the Hermitian form ω on N satisfies only $\partial\bar\partial\omega^{n-1} = 0$. This is because

$$\deg_\omega(E) = \int_N c_1(E,h) \wedge \omega^{n-1}$$

and the difference of any two first Chern forms $c_1(E,h) - c_1(E,h')$ is $\partial\bar\partial$ of a function on N. But then Gauduchon has shown that such an ω exists in the conformal class of every Hermitian metric on N [9, 10]. (Such a metric on N is thus called a Gauduchon metric.) The book of Lübke-Teleman [18] is quite useful, in that it contains most of the relevant theory in one place.

An affine manifold is a real manifold M which admits a flat, torsion-free connection D on its tangent bundle. It is well known (see e.g. [20]) that M is an affine manifold if and only if M admits an affine atlas whose transition functions are locally constant elements of the affine group

$$\mathrm{Aff}(n) = \{\Phi\colon \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi\colon x \mapsto Ax + b\}.$$

(In this case, geodesics of D are straight line segments in the coordinate patches of M.) The tangent bundle TM of an affine manifold admits a natural complex structure, and it is often fruitful to think of M as a real slice of a complex manifold. In particular, local coordinates $x = (x^1, \dots, x^n)$ on M induce the local frame $y = (y^1, \dots, y^n)$ on TM so that every tangent vector y can be written as $y = y^i \frac{\partial}{\partial x^i}$. Then $z^i = x^i + \sqrt{-1}y^i$ form holomorphic coordinates on TM. We will usually denote the complex manifold TM as $M^{\mathbb{C}}$.

Cheng-Yau [4] proved the existence of affine Kähler-Einstein metrics on appropriate affine flat manifolds. The setting in this case is that of affine Kähler, or Hessian, metrics (see also Delanoë [6] for related results). A Riemannian metric g on M is *affine Kähler* if each point has a neighborhood on which there are affine coordinates $\{x^i\}$ and a real potential function ϕ satisfying

$$g_{ij}\, dx^i dx^j = \frac{\partial^2 \phi}{\partial x^i \partial x^j}\, dx^i dx^j.$$

Every Riemannian metric g on M extends to a Hermitian metric $g_{ij}\, dz^i \overline{dz^j}$ on TM. The induced metric on $M^{\mathbb{C}}$ is Kähler if and only if the original metric is affine Kähler.

An important class of affine manifolds is the class of *special affine manifolds*, those which admit a D-covariant constant volume form ν. If such an affine manifold admits an affine Kähler metric, then Cheng-Yau showed that the metric can be deformed to a flat metric by adding the Hessian of a smooth function [4]. There is also the famous conjecture of Markus: A compact affine manifold admits a covariant-constant volume form if and only if D is complete. In the present work, we will use a covariant-constant volume form to convert $2n$-forms on the complex manifold $TM = M^{\mathbb{C}}$ to n-forms on M which can be integrated. The fact that $D\nu = 0$ will ensure that ν does not provide additional curvature terms when integrating by parts on M.

The correct analog of a holomorphic vector bundle over a complex manifold is a flat vector bundle over an affine manifold. In particular, the transition functions of a real vector bundle over an affine flat M may be extended to transition functions on TM by making them constant along the fibers of $M^{\mathbb{C}} \to M$. In the local coordinates as above, we require the transition functions to be constant in the y variables. Such a transition function f is holomorphic over TM exactly when

$$0 = \bar\partial f = \frac{\partial f}{\partial \overline{z^i}}\, \overline{dz^i} = \left(\frac{1}{2}\frac{\partial f}{\partial x^i} + \frac{\sqrt{-1}}{2}\frac{\partial f}{\partial y^i}\right)\overline{dz^i} = \frac{1}{2}\frac{\partial f}{\partial x^i}\,\overline{dz^i},$$

in other words, when the transition function is constant in x. In this way, from any locally constant vector bundle $E \to M$, we can produce a locally constant holomorphic vector bundle of the same rank $E^{\mathbb{C}} \to M^{\mathbb{C}}$.

The existence of Hermitian-Einstein metrics on holomorphic vector bundles over Gauduchon surfaces has been used by Li-Yau-Zheng [16, 17], and also Teleman [23], based on ideas in [16], to provide a new proof of Bogomolov's theorem on compact complex surfaces in Kodaira's class VII_0. Teleman has recently extended these techniques to classify surfaces of class VII with $b_2 = 1$ [22].

The theory we present below is explicitly modeled on Uhlenbeck-Yau and Li-Yau's arguments. We have found it useful to follow the treatment of Lübke-Teleman [18] fairly closely, since most of the relevant theory for Hermitian-Einstein metrics on Gauduchon manifolds is contained in [18]. Our main theorem is

Theorem 1. *Let M be a compact special affine manifold without boundary equipped with an affine Gauduchon metric g. Let $E \to M$ be a flat complex vector bundle. If E is g-stable, then there is an affine Hermitian-Einstein metric on E.*

A similar result holds for flat real vector bundles over M (see Corollary 33 below).

We should remark that the affine Kähler-Einstein metrics produced by Cheng-Yau in [4] are examples of affine Hermitian-Einstein metrics as well: The affine Kähler-Einstein metric g on the affine manifold M can be thought of as a metric on the flat vector bundle TM, and as such a bundle metric, g is affine Hermitian-Einstein with respect to g itself as an affine Kähler metric on M. Cheng-Yau's method of proof is to solve real Monge-Ampère equations on affine manifolds (and they also provide one of the first solutions to the real Monge-Ampère equation on convex domains in [4]).

It is worth pointing out, in broad strokes, how to relate the proof we present below to the complex case: The complex case relies on most of the standard tools of elliptic theory on compact manifolds: the maximum principle, integration by parts, L^p estimates, Sobolev embedding, spectral theory of elliptic operators, and some intricate local calculations. The main innovation we provide to the affine case is Proposition

3 below, which secures our ability to integrate by parts on a special affine manifold. Moreover, by extending a complex flat vector bundle $E \to M$ to a flat holomorphic vector bundle $E^{\mathbb{C}} \to M^{\mathbb{C}}$ as above, we can ensure that the local calculations on M are *exactly the same* as those on $M^{\mathbb{C}}$, and thus we do not have to change these calculations at all to use them in our proof. The maximum principle and spectral theory work the same way in our setting as well. The L^p and Sobolev theories in the complex case do not strongly use the ambient real dimension $2n$ of the complex manifold: and in fact, reducing the dimension to n helps matters.

There are a few other small differences in our approach on affine manifolds as compared to the case of complex manifolds: First of all, we are able to avoid the intricate proof of Uhlenbeck-Yau [24, 25] that a weakly holomorphic subbundle of a holomorphic vector bundle on a complex manifold is a reflexive analytic subsheaf (see also Popovici [19]). The corresponding fact we must prove is that a weakly flat subbundle of a flat vector bundle on an affine manifold is in fact a flat subbundle. We are able to give a quite simple regularity proof in the affine case below in Proposition 27, and the flat subbundle we produce is smooth.

Another small difference between the present case and the complex case concerns simple bundles. The important estimate Proposition 14 below works only for simple bundles E (bundles whose only endomorphisms are multiples of the identity). This does not affect the main theorem in the complex case, for Kobayashi [12] has shown that any stable holomorphic vector bundle over a compact Gauduchon manifold must be simple. For a flat *real* vector bundle E over an affine manifold, there are two possible notions of simple, depending on whether we require every real locally constant section of $\mathrm{End}(E)$ ($\mathbb{R}$-simple), or every complex locally constant section of $\mathrm{End}(E) \otimes_{\mathbb{R}} \mathbb{C}$ ($\mathbb{C}$-simple), to be a multiple of the identity. Since Kobayashi's proof relies on taking an eigenvalue, we must do a little more work in Section 11 below to address the case of $\mathbb{R}$-simple bundles.

In Sections 2 and 3 below, we develop some of the basic theory of (p,q) forms with values in a flat vector bundle E over M, affine Hermitian connections, and the second fundamental form. The basic principle behind these definitions is to mimic the same formulas of the holomorphic vector bundle $E^{\mathbb{C}} \to M^{\mathbb{C}}$. One interesting side note in this story is Lemma 1, which notes for a metric on a real flat vector bundle (E, ∇) over M, the dual connection ∇^* on E is equivalent to the Hermitian connection on $E^{\mathbb{C}} \to M^{\mathbb{C}}$.

Section 4 contains our main technical tool, which allows us to integrate (p, q) forms by parts on a special affine manifold. Then in Section 5, we prove the easy parts of the theory of affine Hermitian-Einstein metrics: vanishing, uniqueness, and stability theorems for affine Hermitian-Einstein metrics, most of which are due to Kobayashi in the complex case. The proofs we present are easier than in the complex case, since we need only consider subbundles, and not singular subsheaves, in our definition of stability. In Section 6, we produce affine Gauduchon metrics on special affine manifolds.

Then in Sections 7 to 10, we prove Theorem 1, following the continuity method of Uhlenbeck-Yau, as modified by Li-Yau for Gauduchon manifolds and as presented in Lübke-Teleman [18]. Since our local calculations are designed to be exactly the same as the complex case, we omit some of these calculations. On the other hand, we do emphasize those parts of the proof which involve integration, as this highlights the main difference between our theory on affine manifolds and the complex case. The regularity result in Section 10 is much easier than that of Uhlenbeck-Yau [24, 25]. Finally, in Section 11, we address the issue of $\mathbb{R}$- and $\mathbb{C}$-simple bundles, to prove a version of the main theorem, Corollary 33 for $\mathbb{R}$-stable flat real vector bundles.

We should also mention Corlette's results on flat principle bundles on Riemannian manifolds:

Theorem 2. [5] *Let G be a semisimple Lie group, (M, g) a compact Riemannian manifold, and P a flat principle G-bundle over M. A* metric *on P is defined to be a reduction of the structure group to K a maximal compact subgroup of G, and a* harmonic metric *is a metric on P so that the induced $\pi_1(M)$-equivariant map from the universal cover $\tilde{M}$ to the Riemannian symmetric space G/K is harmonic. Then P admits a harmonic metric if and only if P is reductive in the sense that the Zariski closure of the holonomy at every point in M is a reductive subgroup of G.*

This theorem is extended to reductive Lie groups G by Labourie [14].

If G is the special linear group, then we may consider the flat vector bundle (E, ∇) associated to P. Then the reductiveness of P is equivalent to the condition on E that any ∇-invariant subbundle has a ∇-invariant complement. For M a compact special affine manifold equipped with an affine Gauduchon metric g, our Theorem 1 produces an affine Hermitian-Einstein bundle metric on a flat vector bundle E when it is slope-stable. If we assume E is irreducible as a flat bundle, then our slope-stability condition is *a priori* weaker than Corlette's: we require every proper flat subbundle of E to have smaller slope, while

Corlette requires that there be no proper flat subbundles of E. It should be interesting to compare the harmonic and affine Hermitian-Einstein metrics on E when they both exist.

It is well known that an affine structure on a manifold M is equivalent to the existence of an *affine-flat* (flat and torsion-free) connection D on the tangent bundle TM, which induces a flat connection on a principle bundle over M with group $G = \mathrm{Aff}(n,\mathbb{R})$ the affine group. The affine group is not semisimple (or even reductive), and so Corlette's result does not apply directly to study this case. On a special affine manifold, however, D induces a flat metric on an n-principal bundle, and Corlette's result applies on TM as a flat n-bundle. Thus, Corlette's result cannot see that D is torsion-free. On the other hand, the affine Hermitian-Einstein metric we produce does essentially use the fact that D is torsion-free: this ensures the induced almost-complex structure on $M^{\mathbb{C}}$ is integrable. So the affine Hermitian-Einstein metrics should be able to exploit the affine structure on M.

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2. Affine Dolbeault complex

On an affine manifold M, there are natural (p, q) forms (see Cheng-Yau [4] or Shima [20]), which are the restrictions of (p, q) forms from $M^{\mathbb{C}}$. We define the space of these forms as

$$\mathcal{A}^{p,q}(M) = \Lambda^p(M) \otimes \Lambda^q(M)$$

for $\Lambda^p(M)$ the usual exterior p-forms on M. If x^i are local affine coordinates on M, then we will denote the induced frame on $\mathcal{A}^{p,q}$ by

$$\{dz^{i_1} \wedge \cdots \wedge dz^{i_p} \otimes d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}\},$$

where we think of $z^i = x^i + \sqrt{-1}\, y^i$ as coordinates on $M^{\mathbb{C}}$ as above.

The flat connection D induces flat connections on the bundles $\Lambda^q(M)$ of q-forms of M. Therefore, the exterior derivative d extends to operators

$$d^D \otimes I \colon \Lambda^p(M) \otimes \Lambda^q(M) \to \Lambda^{p+1}(M) \otimes \Lambda^q(M),$$
$$I \otimes d^D \colon \Lambda^p(M) \otimes \Lambda^q(M) \to \Lambda^p(M) \otimes \Lambda^{q+1}(M).$$

for I the identity operator and d^D the exterior derivative for bundle-valued forms induced from D. These operators are equivalent to the operators ∂ and $\bar{\partial}$ restricted from $M^{\mathbb{C}}$. We find it useful to use the exact restrictions of ∂ and $\bar{\partial}$ (so that, insofar as possible, all the local calculations we do are the same as in the case of complex manifolds). The proper correspondences are, for ∂ and $\bar\partial$ acting on (p,q) forms,

$$\partial = \tfrac{1}{2}(d^D \otimes I), \qquad \bar\partial = (-1)^p \tfrac{1}{2}(I \otimes d^D).$$

A Riemannian metric g on M gives rise to a natural $(1,1)$ form given in local coordinates by $\omega_g = g_{ij} dz^i \otimes d\bar z^j$. This is of course the restriction of the Hermitian form induced by the extension of g to $M^{\mathbb{C}}$.

There is also a natural wedge product on $\mathcal{A}^{p,q}$, which we take to be the restriction of the wedge product on $M^{\mathbb{C}}$: If $\phi_i \otimes \psi_i \in \mathcal{A}^{p_i,q_i}$ for $i = 1, 2$, then we define

$$(\phi_1 \otimes \psi_1) \wedge (\phi_2 \otimes \psi_2) = (-1)^{q_1 p_2} (\phi_1 \wedge \phi_2) \otimes (\psi_1 \wedge \psi_2) \in \mathcal{A}^{p_1+p_2, q_1+q_2}.$$

Consider the space of (p,q) forms $\mathcal{A}^{p,q}(E)$ taking values in a complex (or real) vector bundle $E \to M$. If ∇ is a flat connection on E, and h is a Hermitian metric on E (positive-definite if E is a real bundle), then we consider the Hermitian connection, or Chern connection, on $E^{\mathbb{C}} \to M^{\mathbb{C}}$. Recall the Hermitian connection is the unique connection on a Hermitian holomorphic vector bundle over a complex manifold which both preserves the Hermitian metric and whose $(0,1)$ part is equal to the natural $\bar\partial$ operator on sections of E. Any locally constant frame $s_1, \dots, s_r$ over $E \to M$ extends to a holomorphic frame over $E^{\mathbb{C}} \to M^{\mathbb{C}}$, where we have the usual formula (see e.g. [13]) for the Hermitian connection: If $h_{\alpha\bar\beta} = h(s_\alpha, s_\beta)$, then the connection form is a $\mathrm{End}E$-valued $(1,0)$ form

$$\theta^\alpha_\beta = h^{\alpha\bar\gamma}\, \partial h_{\beta\bar\gamma}.$$

In passing from (p,q) forms on $M^{\mathbb{C}}$ to (p,q) forms on M, we use the following convention:

$$dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_q} \mapsto dz^{i_1} \wedge \cdots \wedge dz^{i_p} \otimes d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_q} \tag{1}$$

As we will see in the next section, this convention will make all the important curvature quantities on $E \to M$ to be real in the case E is a real vector bundle equipped with a real positive-definite metric.

There is also a natural map from (p,q) forms on M to (q,p) forms on M, which is the restriction of complex conjugation on $M^{\mathbb{C}}$: If $\alpha \in \Lambda^p(M)$, $\beta \in \Lambda^q(M)$ are complex valued forms, then we define

$$\overline{\alpha \otimes \beta} = (-1)^{pq} \bar\beta \otimes \bar\alpha. \tag{2}$$

At least when E is a real bundle and h is a real positive-definite metric, the Hermitian connection described above, when restricted to M, has an interpretation in terms of the dual connection of ∇ with respect to h. Recall that the dual connection ∇^* is defined on $E \to M$ by

$$d[h(s_1, s_2)] = h(\nabla s_1, s_2) + h(s_1, \nabla^* s_2)$$

(see e.g. [1]). Then we may define operators $\partial^{\nabla,h}$ and $\bar\partial^{\nabla}$ on $\mathcal{A}^{p,q}(E)$ as follows: For $\phi \in \mathcal{A}^{p,q}$ and $\sigma \in \Gamma(E)$,

$$\begin{aligned} \partial^{\nabla,h}\sigma &= \nabla^*\sigma \otimes \tfrac{1}{2}, \\ \bar\partial^{\nabla}\sigma &= \tfrac{1}{2} \otimes \nabla\sigma, \\ \partial^{\nabla,h}(\sigma \cdot \phi) &= (\partial^{\nabla,h}\sigma) \wedge \phi + \sigma \cdot \partial\phi, \\ \bar\partial^{\nabla}(\sigma \cdot \phi) &= (\bar\partial^{\nabla}\sigma) \wedge \phi + \sigma \cdot \bar\partial\phi. \end{aligned}$$

On M, we consider the pair $(\partial^{\nabla,h}, \bar\partial^{\nabla})$ to form an *extended Hermitian connection* on E, and the extended connection is equivalent to the Hermitian connection on $E^{\mathbb{C}} \to M^{\mathbb{C}}$: The Hermitian connection on $E^{\mathbb{C}} \to M^{\mathbb{C}}$ is given by $d^{\nabla,h} = \partial^{\nabla,h} + \bar\partial^{\nabla} \colon \Lambda^0(E^{\mathbb{C}}) \to \Lambda^1(E^{\mathbb{C}})$.

Also note that the difference $\nabla^* - \nabla$ is a section of $\Lambda^1(\mathrm{End}E)$. This is a similar construction to the first Koszul form on a Hessian manifold (see e.g. Shima [20]).

We have the following lemma, whose proof is a simple computation:

Lemma 1. *If (E, ∇) is a flat real vector bundle over an affine manifold M, and E is equipped with a positive-definite metric h, then the extended Hermitian connection on E (when considered as a complex vector bundle with Hermitian metric induced from h) is given by*

$$(\partial^{\nabla,h}, \bar\partial^{\nabla}) = (\nabla^* \otimes \tfrac{1}{2}, \tfrac{1}{2} \otimes \nabla)$$

for ∇^* the dual connection of ∇ on E with respect to the metric h.

The curvature form $\Omega \in \mathcal{A}^{1,1}(\mathrm{End}E)$ is given by

$$\Omega^\alpha_\beta = \bar\partial\theta^\alpha_\beta = -h^{\alpha\bar\eta}\partial\bar\partial h_{\beta\bar\eta} + h^{\alpha\bar\zeta}h^{\epsilon\bar\eta}\partial h_{\beta\bar\eta} \wedge \bar\partial h_{\epsilon\bar\zeta}.$$

If we write $\Omega^\alpha_\beta = R^\alpha_{\beta i\bar j}\, dz^i \wedge d\bar z^j$, then

$$R^\alpha_{\beta i\bar j} = -h^{\alpha\bar\eta}\frac{\partial^2 h_{\beta\bar\eta}}{\partial z^i \partial\bar z^j} + h^{\alpha\bar\zeta}h^{\epsilon\bar\eta}\frac{\partial h_{\beta\bar\eta}}{\partial z^i}\frac{\partial h_{\epsilon\bar\zeta}}{\partial\bar z^j}.$$

These same formulas represent the restriction of the curvature form of $E^{\mathbb{C}} \to M^{\mathbb{C}}$ to M. On M, we call this the *extended curvature form* (and we still use the symbols $dz^i, d\bar z^j$ to represent elements of $\mathcal{A}^{p,q}$ on M).

We use a Riemannian metric g on M to contract the $(1,1)$ part of an extended curvature form to form a section of $\mathrm{End}E = E^* \otimes E$ which we call the *extended mean curvature*. A metric on E is said to be *affine Hermitian-Einstein* with respect to g if its extended mean curvature

K^α_β is a constant γ times the identity endomorphism of E. In index notation, we have

$$K^\alpha_\beta = g^{i\bar{j}} R^\alpha_{\beta i\bar{j}} = \gamma\, I^\alpha_\beta.$$

(Here we extend the Riemannian metric g to a Hermitian metric $g_{i\bar{j}}$ on $M^{\mathbb{C}}$, and I is the identity endomorphism on E.)

Given a Hermitian locally constant bundle (E, h) on M, the trace $R^\alpha_{\alpha i\bar{j}}$ is called the *extended first Chern form*, or extended Ricci curvature. This first Chern form is give by

$$c_1(E, h) = -\partial\bar{\partial} \log \det h_{\alpha\bar{\beta}},$$

and it may naturally be thought of as the extended curvature of the locally constant line bundle $\det E$ with metric $\det h$.

The extended first Chern form and the extended mean curvature are related by

$$(\operatorname{tr} K)\, \omega_g^n = n\, c_1(E, h) \wedge \omega_g^{n-1}. \tag{3}$$

3. Flat vector bundles

In this section, we collect some facts about flat vector bundles, and representations of the fundamental group, and vector-bundle second fundamental forms. The field $\mathbb{K}$ will represent either $\mathbb{R}$ or $\mathbb{C}$.

A section s of a flat vector bundle (E, ∇) over a manifold M is called *locally constant* if $\nabla s = 0$. Every flat vector bundle has local frames of locally constant sections, given by parallel transport from a basis of a fiber E_x for $x \in M$. For this reason, flat vector bundles are sometimes referred to as locally constant vector bundles.

A flat $\mathbb{K}$-vector bundle of rank r naturally corresponds to a representation ρ of fundamental group into $\mathbf{GL}(r, \mathbb{K})$. For $\tilde{M}$ the universal cover of M, the fundamental group $\pi_1(M)$ acts on total space $\tilde{M} \times \mathbb{K}^r$ equivariantly with respect to the action

$$\gamma \colon (x, y) \to (\gamma(x), \rho(\gamma)(y)).$$

In this picture, a flat subbundle of rank r' is given by an inclusion $\tilde{M} \times \mathbb{K}^{r'} \subset \tilde{M} \times \mathbb{K}^r$ as trivial bundles, where π_1 acts on $\tilde{M} \times \mathbb{K}^{r'}$. In other words, we require for every $\gamma \in \pi_1$ and $y \in \mathbb{R}^{r'}$, $\rho(\gamma)(y) \in \mathbb{R}^{r'}$.

Let (E, ∇) be a flat complex vector bundle over an affine manifold M, and h is a Hermitian metric on E. The geometry of flat subbundles of E follows as in the case of holomorphic bundles on complex manifolds. Let F be a flat subbundle of E (i.e. F is a smooth subbundle of E whose sections s satisfy $\nabla_X s$ is again a section of F for every vector field X on M). Then for any section s of F, we may split $\partial^{\nabla,h}s$ into a part in F and a part h-orthogonal to F:

$$\partial^{\nabla,h}s = \partial^{\nabla_F,h_F}s + A(s).$$

As the notation suggests, the first term on the right $\partial^{\nabla_F,h_F}s$ is the $(1,0)$ part of the affine Hermitian connection induced on F by ∇ and h. The second term A is a $\mathrm{Hom}(F,F^\perp)$-valued $(1,0)$ form called the *second fundamental form* of the subbundle F of E. Note we only need consider $\partial^{\nabla,h}s$ since the second fundamental form is of $(1,0)$ type in the complex case. We have the following proposition (see e.g. [13, Proposition I.6.14])

Proposition 2. *Given (E,∇), h, F and A as above, A vanishes identically if and only if $F^\perp$ is a flat vector subbundle of (E,∇) and the orthogonal decomposition*

$$E = F \oplus F^\perp$$

is a direct sum of flat vector bundles.

4. Integration by parts

The main difference we will discuss between complex and affine manifolds is in integration theory. On an n-dimensional complex manifold, an (n,n) form is a volume form which can be integrated, while on an affine manifold, an (n,n) form is not a volume form. Here we make a crucial extra assumption to handle this case adequately: We assume our affine manifold M is equipped with a D-invariant volume form ν. Equivalently, we assume the linear part of the holonomy of D lies in $\mathbf{SL}(n,\mathbb{R})$. We call such an affine manifold (M,D,ν) a *special affine manifold*. This important special case of affine manifold is quite commonly used: in Strominger-Yau-Zaslow's conjecture [21], a Calabi-Yau manifold N near the large complex structure limit in moduli should be the total space of a (possibly singular) fibration with fibers of special Lagrangian tori over a base manifold which is special affine. (The D-invariant volume form is the restriction of the holomorphic $(n,0)$ form on N.) Also, a famous conjecture of Markus states that a compact affine manifold (M,D) admits a D-invariant volume form if and only if D is complete.

From now on, we assume that M admits a D-invariant volume form ν. Then ν provides natural maps from

$$\mathcal{A}^{n,p} \to \Lambda^p, \qquad \nu \otimes \chi \mapsto (-1)^{\frac{n(n-1)}{2}}\chi;$$
$$\mathcal{A}^{p,n} \to \Lambda^p, \qquad \chi \otimes \nu \mapsto (-1)^{\frac{n(n-1)}{2}}\chi.$$

(The choice of sign is to ensure that for every Riemannian metric g, ω_g^n/ν has the same orientation as ν.) We use division by ν to denote both of these maps. In particular, $\chi \in \mathcal{A}^{n,n}$ can be integrated on M by considering

$$\int_M \frac{\chi}{\nu}.$$

The reason we require ν to be D-invariant is to allow the usual integration by parts formulas for (p,q) forms to work on the affine manifold M. The main result we need is the following:

Proposition 3. *Suppose (M, D) is an affine flat manifold equipped with a D-invariant volume form ν. Then if $\chi \in \mathcal{A}^{n-1,n}$,*

$$\frac{\partial\chi}{\nu} = d\left(\frac{\chi}{2\nu}\right).$$

Also, if $\chi \in \mathcal{A}^{n,n-1}$,

$$\frac{\bar\partial\chi}{\nu} = (-1)^n\, d\left(\frac{\chi}{2\nu}\right).$$

Proof. We may choose local affine coordinates $x^1, \dots, x^n$ on M so that $\nu = dx^1 \wedge \cdots \wedge dx^n$, and write $\chi \in \mathcal{A}^{n-1,n}$ locally as

$$\begin{aligned}
\chi &= \sum_{i=1}^n f_i\, dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge dz^n \otimes d\bar z^1 \wedge \cdots \wedge d\bar z^n, \\
\partial\chi &= \frac{1}{2}\sum_{i=1}^n (-1)^{i-1}\frac{\partial f_i}{\partial x^i}\, dz^1 \wedge \cdots \wedge dz^n \otimes d\bar z^1 \wedge \cdots \wedge d\bar z^n, \\
\frac{\chi}{\nu} &= (-1)^{\frac{n(n-1)}{2}} \sum_{i=1}^n f_i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \\
d\left(\frac{\chi}{\nu}\right) &= (-1)^{\frac{n(n-1)}{2}} \sum_{i=1}^n (-1)^{i-1}\frac{\partial f_i}{\partial x^i}\, dx^1 \wedge \cdots \wedge dx^n.
\end{aligned}$$

(Note that when restricted to M, $dz^i = d\bar z^i = dx^i$.) The computation is similar for $\chi \in \mathcal{A}^{n,n-1}$. □

To each Riemannian metric g on an affine manifold M, there is a natural nondegenerate $(1,1)$ form given by $\omega_g = g_{ij}\, dx^i \otimes dx^j$ for x^i local coordinates on M. (The metric g is naturally extended to a Hermitian metric on $M^{\mathbb{C}}$ and ω_g is the restriction of the Hermitian form of g to $M \subset M^{\mathbb{C}}$.) A metric g on M is said to be *affine Gauduchon* if $\partial\bar\partial(\omega_g^{n-1}) = 0$. We will see in the next section that every conformal class of Riemannian metrics on a compact special affine manifold M contains an affine Gauduchon metric.

Note that by our convention (1) our definition of first Chern form is a *real* $(1,1)$ form on M, even though it is the restriction of an *imaginary* 2 form on $M^{\mathbb{C}}$.

A locally constant vector bundle E over a special affine manifold (M,ν) equipped with an affine Gauduchon metric g has a *degree* given by

$$\deg_g E = \int_M \frac{c_1(E,h)\wedge \omega_g^{n-1}}{\nu}. \tag{4}$$

Recall the affine first Chern form $c_1(E,h) = -\partial\bar\partial \log\det h_{\alpha\bar\beta}$ for any Hermitian metric h on E. The degree is well-defined because

- For any other metric h' on E,
  $$c_1(E,h') - c_1(E,h) = \partial\bar\partial(\log\det h_{\alpha\bar\beta} - \log\det h'_{\alpha\bar\beta}),$$
  which is $\partial\bar\partial$ of a function on M.
- Proposition 3 above allows us to integrate by parts to move the $\partial\bar\partial$ to ω_g^{n-1}.
- The metric g is affine Gauduchon.

Note we do not expect the degree to be an integer (see e.g. Lübke-Teleman [18] for counterexamples in the complex case).

The *slope* of a flat vector bundle E over a special affine manifold M equipped with an affine Gauduchon metric g is defined to be

$$\mu_g = \frac{\deg_g E}{\operatorname{rank} E}.$$

Such a complex flat vector bundle E is called $\mathbb{C}$-*stable* if every flat subbundle F of E satisfies

$$\mu_g(F) < \mu_g(E). \tag{5}$$

A real flat vector bundle E is called $\mathbb{R}$-*stable* if (5) is satisfied by any flat real vector subbundle F of E, while such an E is called $\mathbb{C}$-stable if the complex vector bundle $E\otimes_{\mathbb{R}}\mathbb{C}$ is $\mathbb{C}$-stable.

5. Affine Hermitian-Einstein metrics

In this section, we will check some of the basic properties of Hermitian-Einstein metrics extend to the affine case: a vanishing theorem of Kobayashi, uniqueness of affine Hermitian-Einstein metrics on simple bundles, and stability of flat bundles admitting affine Hermitian-Einstein metrics. These roughly form the easy part of the Kobayashi-Hitchin correspondence between stable bundles and Hermitian-Einstein metrics. The hard part, to prove the existence of Hermitian-Einstein metrics, will be addressed in the Sections 7 to 10 below.

We have the following vanishing theorem of Kobayashi [13]

Theorem 3. *Let (E, ∇) be a flat vector bundle over a compact affine manifold M equipped with a Riemannian metric g. Assume E admits an affine Hermitian-Einstein metric h with Einstein factor γ_h. If $\gamma_h < 0$, then E has no nontrivial locally constant sections. If $\gamma_h = 0$, then any locally constant section s of E satisfies $\partial^h s = 0$ for $\partial^h = \partial^{\nabla,h}$.*

Proof. For s any locally constant section of E, compute

$$\mathrm{tr}_g \partial\bar{\partial}|s|^2 = -\gamma_h |s|^2 + |\partial^h s|^2$$

and apply the maximum principle. □

The following uniqueness proposition follows Lübke-Teleman [18, Prop. 2.2.2]

Proposition 4. *If (E, ∇) is a simple flat vector bundle over a compact affine manifold M with Riemannian metric g, then any g-affine-Hermitian-Einstein metric on E is unique up to a positive scalar.*

Proof. Let h_1, h_2 be two affine Hermitian-Einstein metrics on E with Einstein constants γ_1, γ_2. Then there an endomorphism f of E satisfying $h_2(s,t) = h_1(f(s),t)$ for all sections s,t, and since h_1, h_2 are both Hermitian, $f^{\frac{1}{2}}$ is well-defined.

Then the connection $\nabla' = f^{\frac{1}{2}} \circ \nabla \circ f^{-\frac{1}{2}}$ is a flat connection on E. Let E' signify the new flat structure ∇' induces on the underlying vector bundle of E, and let E signify the original flat structure ∇. Then $f^{\frac{1}{2}}$ is a locally constant section of the flat vector bundle $\mathrm{Hom}(E, E')$, h_1 is affine Hermitian-Einstein with Einstein constant γ_2 on E', and so the metric induced on $\mathrm{Hom}(E, E')$ by h_1 on E' and h_2 on E is affine Hermitian-Einstein with Einstein constant $\gamma_2 - \gamma_2 = 0$.

Therefore, Theorem 3 applies, to show that $\partial_{\mathrm{Hom}}(f^{\frac{1}{2}}) = 0$ for ∂_{Hom} the $(1,0)$ part of the affine Hermitian-Einstein connection on $\mathrm{Hom}(E, E')$. A computation as in [18] then implies that $\partial_1 f = 0$ for ∂_1 the $(1,0)$ part of the affine Hermitian connection on (E, h_1). Since f is h_1-self-adjoint, this implies $\bar{\partial}(f^*) = \bar{\partial} f = 0$.

So since (E, ∇) is simple, f is a multiple of the identity. □

The following theorem is due to Kobayashi in the Kähler case [13]. The proof in the present case is simpler because we need only deal with subbundles and not singular subsheaves in the definition of stability.

Theorem 4. *Let E be a flat vector bundle over a compact special affine manifold M equipped with an affine Gauduchon metric g. If E admits an affine Hermitian-Einstein metric h with Einstein constant γ, then* *either E is g-stable or E is an h-orthogonal direct sum of flat stable vector bundles, each of which is affine Hermitian-Einstein with Einstein constant γ.*

Proof. Consider E' a flat subbundle of E. Then it suffices to prove that $\mu(E') \leq \mu(E)$ with equality only in the case that the h-orthogonal complement of $E' \subset E$ is also a flat subbundle of E. By Proposition 2 above, it suffices to show that $\mu(E') \leq \mu(E)$ with equality only if the second fundamental form of $E' \subset E$ vanishes.

We compute, as in [18, Proposition 2.3.1] or [13, Proposition V.8.2] that for $s = \operatorname{rank} E'$, $r = \operatorname{rank} E$, that

$$\begin{aligned} \mu_g E &= \frac{1}{rn} \int_M \operatorname{tr} K_E \frac{\omega_g^n}{\nu} \\ &= \frac{\gamma}{n} \int_M \frac{\omega_g^n}{\nu}, \\ \mu_g E' &= \frac{1}{sn} \int_M \operatorname{tr} K_{E'} \frac{\omega_g^n}{\nu} \\ &= \frac{\gamma}{n} \int_M \frac{\omega_g^n}{\nu} - \frac{1}{sn} \int_M |A|^2 \frac{\omega_g^n}{\nu}. \end{aligned}$$

Thus $\mu_g E \leq \mu_g E'$ always, with equality if and only if the second fundamental form A is identically 0.

For the exact sequence of flat bundles $0 \to E' \to E \to E'' \to 0$, the extended curvatures R', R, and R'' of the Hermitian connections induced by h on E', E, E'' respectively, satisfy

$$R = \begin{pmatrix} R' + A \wedge A^* & * \\ * & R'' + A^* \wedge A \end{pmatrix}$$

(see e.g. Kobayashi [13, Proposition I.6.14]). So the vanishing of A implies that the mean curvatures $K' = \operatorname{tr}_g R'$ of E' and $K'' = \operatorname{tr}_g R''$ of E'' satisfy the Hermitian-Einstein condition if K does. Thus, in the case of equality $\mu_g E = \mu_g E'$, E splits into proper flat affine Hermitian-Einstein summands E' and E''. The theorem then follows by induction on the rank r. □

6. Affine Gauduchon metrics

Given a smooth Riemannian metric g on an affine manifold M with parallel volume form ν, define the operator from functions to functions given by

$$Q(\phi) = \frac{\partial\bar\partial(\phi\, \omega_g^{n-1})}{\omega_g^n}. \tag{6}$$

If we can find a smooth, positive solution to $Q(\phi)=0$, then $\phi^{\frac{1}{n-1}}g$ is affine Gauduchon.

Consider the adjoint Q^* of Q with respect to the inner product

$$\langle \phi, \psi\rangle_g = \int_M \phi\,\psi \frac{\omega_g^n}{\nu}. \tag{7}$$

Note that we are *not* integrating with respect to the volume form of g. We can avoid extra curvature terms by using the volume form ω_g^n/ν instead (these terms are worked out in the case of affine Kähler manifolds by Shima [20]). Compute, using Proposition 3 above,

$$\begin{aligned} \langle \phi, Q^*(\psi)\rangle_g &= \langle Q(\phi),\psi\rangle_g \\ &= \int_M \frac{\partial\bar\partial(\phi\,\omega_g^{n-1})}{\omega_g^n}\,\psi\,\frac{\omega_g^n}{\nu} \\ &= \int_M \phi\,\frac{\partial\bar\partial\psi\wedge\omega_g^{n-1}}{\nu}, \\ Q^*(\psi) &= \frac{\partial\bar\partial\psi\wedge\omega_g^{n-1}}{\omega_g^n} \\ &= \frac{1}{4n}\,g^{ij}\frac{\partial^2\psi}{\partial x^i\partial x^j} = \frac{1}{n}\mathrm{tr}_g\partial\bar\partial\psi. \end{aligned}$$

We have the following lemma

Lemma 5. *The kernel of Q^* consists of only the constant functions. The only nonnegative function in the image of Q^* is the zero function.*

Proof. Both statements follow directly from the strong maximum principle. □

The index of Q (and of Q^*) is 0, as it is an elliptic second-order operator on functions. The previous lemma shows the kernel of Q^* is one-dimensional, and thus the cokernel of Q^* (which may be identified with the kernel of Q by orthogonal projection) is one-dimensional as well. We want to exhibit a positive function in the one-dimensional space $\ker Q$.

Let $\phi\in\ker Q$ be not identically zero. If ψ is not in the image of Q^*, then $\langle\phi,\psi\rangle_g\neq 0$. This is because the dimension of the cokernel of Q^* is one, and the functional

$$\psi\mapsto\langle\phi,\psi\rangle_g$$

is not identically zero but is zero on the image of Q^*. If ϕ assumes both positive and negative values, then we can find a positive function ψ on M so that $\langle\phi,\psi\rangle_g=0$. But Lemma 5 above shows this ψ is not in the range of Q^*, a contradiction. Therefore, ϕ does not assume both positive and negative values. Assume without loss of generality that $\phi \geq 0$.

Now, since $\phi \in \ker Q$ is not identically zero, and since Q is an elliptic linear operator, the strong maximum principle shows that $\phi > 0$. C^∞ regularity of ϕ is standard. So the above discussion has proved

Theorem 5. *If M is a compact affine manifold with covariant-constant volume form ν, then every conformal class of Riemannian metrics on M contains an affine Gauduchon metric unique up to scaling by a constant.*

We will need the following lemma later.

Lemma 6. *If g is an affine Gauduchon metric on a compact special affine manifold, then the kernel of Q consists only of the constant functions.*

Proof. If $\partial\bar\partial\omega_g^{n-1} = 0$, then the definition (6) shows that in local affine coordinates, Q is an elliptic operator of the form

$$Q(\phi) = a^{ij}\frac{\partial^2\phi}{\partial x^i \partial x^j} + b^j \frac{\partial \phi}{\partial x^j}.$$

So the strong maximum principle applies, and any function in the kernel of Q must be constant. □

7. The continuity method

Consider a compact affine manifold M equipped with a covariant-constant volume form ν and an affine Gauduchon metric g, and a flat complex vector bundle E over M, together with a Hermitian metric h_0. Let K_0 be the extended mean curvature of (E, h_0). Equations (3) and (4) show

$$\int_M (\operatorname{tr} K_0)\,\frac{\omega_g^n}{\nu} = n \deg_g E,$$

and therefore for any affine Hermitian-Einstein metric on E (satisfying $K = \gamma\, I_E$), γ must satisfy

$$\gamma \int_M \frac{\omega_g^n}{\nu} = n\,\frac{\deg_g E}{\operatorname{rank} E} = n\,\mu_g E. \tag{8}$$

Let h_0 be a background Hermitian metric E. Then any other Hermitian metric h on E is given may be represented by an endomorphism f of E, so that for sections s, t,

$$h(s,t) = h_0(f(s),t) \qquad \Longleftrightarrow \qquad f_\alpha^\eta = h_0^{\eta\bar\beta} h_{\alpha\bar\beta}.$$

The new metric h is Hermitian if and only if f is Hermitian self-adjoint and positive with respect to h_0. Here are some standard formulas for how the extended connection form θ, curvature Ω, first Chern form c_1 and mean curvature K change when passing from h_0 to h:

$$\theta = \theta_0 + f^{-1}\partial_0 f, \tag{9}$$

$$\Omega = \bar\partial\theta = \Omega_0 + \bar\partial(f^{-1}\partial_0 f), \tag{10}$$

$$K = K_0 + \mathrm{tr}_g[\bar\partial(f^{-1}\partial_0 f)], \tag{11}$$

$$c_1(E,h) = c_1(E,h_0) - \partial\bar\partial \log\det f, \tag{12}$$

$$\mathrm{tr}\, K = \mathrm{tr}\, K_0 - \mathrm{tr}_g \partial\bar\partial(\log\det f). \tag{13}$$

Note that in a locally constant frame, $f^{-1}\partial_0 f$ may be written as $(f^{-1})^\alpha_\eta(\partial_0 f)^\eta_\beta$. The term $\partial_0 f$ is the extended Hermitian connection induced from (E,h_0) onto $\mathrm{End}\, E$ acting on f:

$$(\partial_0 f)^\alpha_\beta = \partial f^\alpha_\beta - (\theta_0)^\eta_\beta f^\alpha_\eta + (\theta_0)^\alpha_\eta f^\eta_\beta.$$

Equation (11) shows that we want to solve the equation

$$K_0 - \gamma\, I_E + \mathrm{tr}_g[\bar\partial(f^{-1}\partial_0 f)] = 0.$$

We will solve this by the continuity method. In particular, for $\epsilon \in [0,1]$, consider the equation

$$L_\epsilon(f) = K_0 - \gamma\, I_E + \mathrm{tr}_g[\bar\partial(f^{-1}\partial_0 f)] + \epsilon\, \log f = 0. \tag{14}$$

Note that since f is an endomorphism of E which is positive Hermitian with respect to h_0, $\log f$ is well-defined.

Assume the background data g and h_0 are smooth. Let

$$J = \{\epsilon \in (0,1] : \text{there is a smooth solution to } L_\epsilon(f) = 0\}.$$

We will use the continuity method to show that $J = (0,1]$ for any $\mathbb{C}$-simple flat vector bundle E, and then later show that we may take $\epsilon \to 0$ if E is $\mathbb{C}$-stable. (If E is $\mathbb{C}$-stable, it is automatically $\mathbb{C}$-simple—see Proposition 30 below.)

The first step in the continuity method is to show $1 \in J$ and so J is nonempty. The proof will also provide an appropriately normalized initial metric h_0 on E.

Proposition 7. *Given a compact special affine manifold M with an affine Gauduchon metric and a flat vector bundle E. Then there is a smooth Hermitian metric h_0 on E so that there is a smooth solution f_1 to $L_1(f) = 0$. The metric h_0 satisfies the normalization $\mathrm{tr}\, K_0 = r\,\gamma$ for r the rank of E and γ given by (8).*

Proof. We first produce the metric h_1 the metric satisfying the L_1 equation, and then we will produce h_0 from h_1.

Given an arbitrary background metric h_0', equation (13) above shows that if $h_1 = e^\rho h_0'$ satisfies $\mathrm{tr}\, K_1 = r\,\gamma$ if and only if

$$\mathrm{tr}_g \partial\bar\partial\rho = \frac{1}{r}\,\mathrm{tr}\, K_0' - \gamma \tag{15}$$

for r the rank of E. Note the right-hand side satisfies

$$\int_M \left(\frac{1}{r}\,\mathrm{tr}\, K_0' - \gamma\right)\frac{\omega_g^n}{\nu} = 0. \tag{16}$$

Lemma 6 shows that the kernel of Q consists of only constants. Equation (16) then shows that the right-hand side of (15) is orthogonal to $\ker Q$ with respect to the inner product (7), and so must be in the image of $Q^* = \frac{1}{n}\mathrm{tr}_g\partial\bar\partial$.

Now define $f_1 = \exp(-K_1 + \gamma I_E)$ and

$$(h_0)_{\alpha\bar\beta} = (f_1^{-1})^\eta_\alpha (h_1)_{\eta\bar\beta}.$$

Then we may check as in Lübke-Teleman [18, Lemma 3.2.1] that h_0 is a Hermitian metric and that, with respect to h_0, f_1 satisfies $L_1(f_1) = 0$. Moreover,

$$\begin{aligned}
\mathrm{tr}\, K_0 &= \mathrm{tr}\, K_1 + \mathrm{tr}_g\partial\bar\partial \log\det f_1 \\
&= \mathrm{tr}\, K_1 + \mathrm{tr}_g\partial\bar\partial\,\mathrm{tr}(-K_1 + \gamma I_E) \\
&= \mathrm{tr}\, K_1 = r\,\gamma.
\end{aligned}$$

□

So for the choice of h_0 derived in Proposition 7, we have

Corollary 8. *$1 \in J$.*

8. Openness

Consider $\mathrm{Herm}(E, h_0)$ to be the space of endomorphisms of the vector bundle E which are Hermitian self-adjoint with respect to h_0. In particular, we may check as in e.g. [18, Lemma 3.2.3] that for f a positive Hermitian endomorphism of E, the operator

$$\hat L(\epsilon, f) = f L_\epsilon(f) = fK - \gamma f + \epsilon f \log f \in \mathrm{Herm}(E, h_0).$$

Let $1 < p < \infty$ and k be a sufficiently large integer.

Assume $\epsilon \in J$—in other words, there is a smooth solution f_ϵ to $L_\epsilon(f) = 0 \iff \hat L(\epsilon, f) = 0$. Then we will use the Implicit Function Theorem to show that there is a $\delta > 0$ so that for every $\epsilon' \in (\epsilon-\delta, \epsilon+\delta)$, there is a solution to $\hat L(\epsilon', f) = 0$ in $L^p_k\mathrm{Herm}(E, h_0)$. Then, for k large enough, we can bootstrap to show C^∞ regularity of each solution $f_{\epsilon'}$ to $\hat{L}(\epsilon', f_{\epsilon'}) = 0$. Thus $(\epsilon - \delta, \epsilon + \delta) \cap (0, 1] \subset J$ and J is open.

So as usual, everything boils down to the checking the hypothesis of the Implicit Function Theorem:

$$\Xi = \frac{\delta}{\delta f}\,\hat{L}(\epsilon, f)\colon L^p_k\mathrm{Herm}(E, h_0) \to L^p_{k-2}\mathrm{Herm}(E, h_0)$$

should be an isomorphism of Banach spaces. The operator $\frac{\delta}{\delta f}\,\hat{L}(\epsilon, f)$ is Fredholm and elliptic. The next thing to check is that the index of the operator Ξ is 0.

Lemma 9. *The index of* Ξ *is 0.*

Proof. To check this, we need only look at the symbol. For $\phi \in \mathrm{Herm}(E, h_0)$, compute

$$\Xi(\phi) \equiv \mathrm{tr}_g \bar{\partial}\partial_0\phi,$$

where $\equiv$ denotes equivalence up to zeroth- and first-order derivatives of ϕ. Moreover, if $\phi, \xi \in \mathrm{Herm}(E, h_0)$, then we may compute

$$\bar{\partial}\,[h_0(\partial_0\phi, \xi)] = h_0(\bar{\partial}\partial_0\phi, \xi) - h_0(\partial_0\phi, \partial_0\xi). \tag{17}$$

Here h_0 acts only on the $\mathrm{End}(E)$ part of the quantities, and not on the differential form parts: For ϕ_1, ϕ_2 sections of $\mathrm{End}(E)$, and $\lambda_i \in \mathcal{A}^{p_i,q_i}$,

$$h_0(\phi_1 \otimes \lambda_1, \phi_2 \otimes \lambda_2) = h_0(\phi_1, \phi_2)\lambda_1 \wedge \bar{\lambda}_2. \tag{18}$$

The ∂_0 in the last time is because of the convention (2) and the fact that h_0 is $\mathbb{C}$-antilinear in the second slot, while the minus sign in front of the last term is because of (18).

Now we use (17) to compute the highest-order terms of the adjoint Ξ^* of Ξ with respect to the inner product

$$\langle \phi, \xi \rangle_{\mathrm{End}(E)} = \int_M h_0(\phi, \xi)\frac{\omega_g^n}{\nu}.$$

Then compute using (17) and Proposition 3:

$$\begin{aligned}
\langle \phi, \Xi^*\xi\rangle_{\mathrm{End}(E)} &= \langle \Xi\phi, \xi\rangle_{\mathrm{End}(E)} \\
&= \int_M h_0(\mathrm{tr}_g\bar\partial\partial_0\phi, \xi)\frac{\omega_g^n}{\nu} \\
&= n\int_M \frac{h_0(\bar\partial\partial_0\phi, \xi)\wedge\omega_g^{n-1}}{\nu} \\
&= -n\int_M \frac{h_0(\partial_0\phi, \partial_0\xi)\wedge\omega_g^{n-1} - h_0(\partial_0\phi, \xi)\wedge\bar\partial\omega_g^{n-1}}{\nu} \\
&= n\int_M \frac{h_0(\phi, \bar\partial\partial_0\xi)\wedge\omega_g^{n-1} + T}{\nu} \\
&= \int_M \frac{h_0(\phi, \mathrm{tr}_g\bar\partial\partial_0\xi)\,\omega_g^n + T}{\nu},
\end{aligned}$$

where T represents terms that involve no derivatives of ϕ and only zeroth- or first-order derivatives of ξ. Therefore, we see

$$\Xi^*(\phi) \equiv \mathrm{tr}_g\bar\partial\partial_0\phi \equiv \Xi(\phi).$$

Since Ξ and Ξ^* have the same symbols, they are homotopic as elliptic operators, and thus have the same index. Since the sum of the indices of Ξ and Ξ^* is 0, they each must have index 0. □

Since the index of Ξ is 0, it suffices to show Ξ is injective to apply the Implicit Function Theorem. In order to do this, we apply the following crucial estimate, essentially due to Uhlenbeck-Yau.

Proposition 10. *Let $\alpha \in \mathbb{R}$, $\epsilon \in (0,1]$, f be a positive and Hermitian endomorphism of E with respect to h_0, and $\phi \in Herm(E, h_0)$. Assume $\hat{L}(\epsilon, f) = 0$ and*

$$\frac{\delta}{\delta f}\hat{L}(\epsilon, f)(\phi) + \alpha f\log f = \Xi(\phi) + \alpha f \log f = 0.$$

Then if $\eta = f^{-\frac{1}{2}}\phi f^{-\frac{1}{2}}$,

$$-tr_g\partial\bar\partial|\eta|^2 + 2\epsilon|\eta|^2 + |\partial_0^f\eta|^2 + |\bar\partial^f\eta|^2 \le -2\alpha h_0(\log f, \eta),$$

where $\partial_0^f = Ad\,f^{-\frac{1}{2}}\circ\partial_0\circ Ad\,f^{\frac{1}{2}}$ and $\bar\partial^f = Ad\,f^{\frac{1}{2}}\circ\bar\partial\circ Ad\,f^{-\frac{1}{2}}$, $|\partial_0^f\eta|^2 = tr_g h_0(\partial_0^f\eta, \partial_0^f)$, and $|\bar\partial^f\eta|^2 = -tr_g h_0(\bar\partial^f\eta, \bar\partial^f\eta)$.

Proof. This is a local calculation on M, which by our definitions of extended Hermitian connections, p, q forms, etc., is the same as the calculation on $M^{\mathbb{C}}$. So we refer the reader to [18, Proposition 3.2.5]. □

Proposition 11. *J is open.*

Proof. By the discussion above, we need only check that Ξ in injective. This follows from the previous Proposition 10 with $\alpha = 0$. In this case,

$$-\mathrm{tr}_g \partial\bar\partial|\eta|^2 + 2\epsilon|\eta|^2 \le 0,$$

and the maximum principle implies $|\eta|^2 = 0$. So $\eta = 0$ and $\phi = 0$. Ξ is injective. □

9. Closedness

Lemma 12. *If f is a Hermitian positive endomorphism of E with respect to h_0 which solve $L_\epsilon(f) = 0$ for $\epsilon > 0$, then* $\det f = 1$.

Proof. Taking the trace of the definition (14) and using Proposition 7, we see that

$$-\mathrm{tr}_g \partial\bar\partial \log\det f + \epsilon \log\det f = 0.$$

The maximum principle then implies $\log\det f = 0$. □

We introduce some more notation. Let $f = f_\epsilon$ represent the family of solutions constructed for ϵ in the interval $(\epsilon_0, 1]$ in Corollary 8 and Proposition 11. Define

$$m = m_\epsilon = \max|\log f_\epsilon|, \quad \phi = \phi_\epsilon = \frac{df_\epsilon}{d\epsilon}, \quad \eta = \eta_\epsilon = f_\epsilon^{-\frac{1}{2}}\phi_\epsilon f_\epsilon^{-\frac{1}{2}}.$$

We can immediately verify

Lemma 13. *The trace* $\mathrm{tr}\,\eta_\epsilon = 0$.

Proof. Compute

$$\mathrm{tr}\,\eta = \mathrm{tr}\,(f^{-\frac{1}{2}}\phi f^{-\frac{1}{2}}) = \mathrm{tr}\left(f^{-1}\frac{df}{d\epsilon}\right) = \frac{d}{d\epsilon}(\log\det f) = 0$$

by Lemma 12 above. □

Proposition 14. *Let E be a $\mathbb{C}$-simple complex flat vector bundle over a compact special affine manifold M. On M, consider the L^2 inner products on $\mathcal{A}^{p,q}(End\,E)$ given by h_0, g and the volume form ω_g^n/ν. Then there is a constant $C(m)$ depending only on M, g, h_0, ν and $m = m_\epsilon$ so that for $\eta = \eta_\epsilon$,*

$$\|\bar\partial^f \eta\|_{L^2}^2 \ge C(m)\|\eta\|_{L^2}^2.$$

Remark. In the following sections, $C(m)$ will denote a constant depending on m and the other objects noted above, but the particular constant may change with the context. C will similarly denote a constant depending only on the initial conditions M, g, h_0 and ν, but not on ϵ or m.

Proof. Let $\psi = f^{-\frac{1}{2}}\eta f^{\frac{1}{2}}$. Then pointwise,

$$|\bar\partial^f\eta|^2 = |f^{\frac{1}{2}}\bar\partial\psi f^{-\frac{1}{2}}|^2 \geq C(m)|\bar\partial\psi|^2.$$

Integrate over M with respect to the volume form ω_g^n/ν to find that

$$\|\bar\partial^f\eta\|^2_{L^2} \geq C(m)\|\bar\partial\psi\|^2_{L^2} = C(m)\langle\bar\partial^*\bar\partial\psi,\psi\rangle,$$

where $\bar\partial^*$ is the adjoint of $\bar\partial$ with respect to the L^2 inner products on $\mathcal{A}^{0,0}(\operatorname{End}E)$ and $\mathcal{A}^{0,1}(\operatorname{End}E)$. It is straightforward to check that $\bar\partial^*\bar\partial\colon \mathcal{A}^{0,0}(\operatorname{End}E)\to\mathcal{A}^{0,0}(\operatorname{End}E)$ is elliptic, and it is self-adjoint by formal properties of the adjoint.

Now $\operatorname{tr}\psi = \operatorname{tr}(f^{-\frac{1}{2}}\eta f^{\frac{1}{2}}) = \operatorname{tr}\eta = 0$, and so for I_E the identity endomorphism of E,

$$\langle\psi, I_E\rangle_{L^2} = \int_M h_0(\psi, I_E)\frac{\omega_g^n}{\nu} = \int_M \operatorname{tr}(\psi I_E)\frac{\omega_g^n}{\nu} = 0,$$

since $h_0(\psi, I_E) = \operatorname{tr}(\psi I_E^*)$ for $I_E^* = I_E$ the adjoint of I_E with respect to h_0. Since E is $\mathbb{C}$-simple, this shows that ψ is L^2-orthogonal to the kernel of $\bar\partial$ on $\operatorname{End}E$. Therefore, since $\bar\partial^*\bar\partial$ is self-adjoint and elliptic, there is a constant $\lambda_1 > 0$ (the smallest positive eigenvalue of $\bar\partial^*\bar\partial$) so that

$$\langle\bar\partial^*\bar\partial\psi,\psi\rangle_{L^2} \geq \lambda_1\|\psi\|^2_{L^2}.$$

Therefore,

$$\|\bar\partial^f\eta\|^2_{L^2} \geq C(m)\langle\bar\partial^*\bar\partial\psi,\psi\rangle_{L^2} \geq C(m)\|\psi\|^2_{L^2} \geq C(m)\|\eta\|^2_{L^2}.$$

□

Now we need the following consequence of a subsolution estimate of Trudinger [11, Theorem 9.20]:

Proposition 15. *If u is a C^2 nonnegative function on M which satisfies*

$$\operatorname{tr}_g\partial\bar\partial u \geq \lambda\, u + \mu$$

for $\lambda \leq 0$ and μ real constants, then

$$\max_M u \leq B(\|u\|_{L^1} + |\mu|)$$

for B a constant only depending on g, ν and λ.

Now we bound $|\phi| = |\phi_\epsilon|$ in terms of m.

Proposition 16. *Given E a $\mathbb{C}$-simple complex flat vector bundle over a compact special affine manifold M, $\max_M|\phi_\epsilon| \leq C(m)$.*

Proof. Proposition 10 above shows that

$$-\mathrm{tr}_g\partial\bar{\partial}|\eta|^2 + |\bar{\partial}^f\eta|^2 \le 2|\log f|\cdot|\eta|.$$

Since $\int_M \mathrm{tr}_g\partial\bar{\partial}|\eta|^2\omega_g^n/\nu = 0$, we have

$$\|\bar{\partial}^f\eta\|_{L^2}^2 \le C(m)\,\|\eta\|_{L^2}.$$

But then Proposition 14 implies

$$C(m)\,\|\eta\|_{L^2}^2 \le \|\bar{\partial}^f\eta\|_{L^2}^2 \le C(m)\,\|\eta\|_{L^2} \qquad \Longrightarrow \qquad \|\eta\|_{L^2} \le C(m).$$

But then we also have from Proposition 10 that

$$-\mathrm{tr}_g\partial\bar{\partial}|\eta|^2 \le 2|\log f|\cdot|\eta| \le m\,|\eta|^2 + m,$$

and Proposition 15 then shows that

$$\max_M |\eta|^2 \le C(m)(\|\eta\|_{L^2}^2 + m) \le C(m).$$

The result follows since $\phi = f^{\frac{1}{2}}\eta f^{\frac{1}{2}}$. □

The following lemma follows is a local calculation as in [18, Lemma 3.3.4.i].

Lemma 17.

$$-\tfrac{1}{2}\mathrm{tr}_g\partial\bar{\partial}|\log f|^2 + \epsilon\,|\log f|^2 \le |K_0 - \gamma I_E|\cdot|\log f|.$$

Corollary 18. $m \le \epsilon^{-1}C$.

Proof. Apply the maximum principle to Lemma 17 for $C = \max_M |K_0 - \gamma I_E|$. □

Corollary 19. $m \le C(\|\log f\|_{L^2} + 1)^2$.

Proof. Lemma 17 implies

$$-\mathrm{tr}_g\partial\bar{\partial}|\log f|^2 \le |\log f|^2 + \max_M |K_0 - \gamma I_E|^2.$$

Then Proposition 15 applies to show

$$m \le C(\|\log f\|_{L^2}^2 + 1),$$

which implies the corollary. □

Lemma 20. *Consider the operator $\bar{\partial}_0^*\bar{\partial}_0$ acting on sections of End(E), where the adjoint is with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathrm{End}(E)}$. Then for each section ψ of End(E),*

$$\partial_0^*\partial_0\psi = \frac{1}{n}\,\mathrm{tr}_g\bar{\partial}\partial_0\psi - \frac{\partial_0\psi\wedge\bar{\partial}\omega_g^{n-1}}{\omega_g^n}.$$

Proof. Since

$$\begin{aligned}\bar\partial[h_0(\partial_0\psi_1,\psi_2)\wedge\omega_g^n] &= [h_0(\bar\partial\partial_0\psi_1,\psi_2)-h_0(\partial_0\psi_1,\partial_0\psi_2)]\wedge\omega_g^{n-1}\\ &\quad -h_0(\partial_0\psi_1,\psi_2)\wedge\bar\partial\omega_g^{n-1},\end{aligned}$$

Proposition 3 and Stokes' Theorem show that

$$\begin{aligned}\int_M h_0(\partial_0^*\partial_0\psi_1,\psi_2)\frac{\omega_g^n}{\nu} &= \int_M \frac{h_0(\partial_0\psi_1,\partial_0\psi_2)\wedge\omega_g^{n-1}}{\nu}\\ &= \int_M \frac{h_0(\bar\partial\partial_0\psi_1,\psi_2)\wedge\omega_g^{n-1}}{\nu}\\ &\quad -\int_M \frac{h_0(\partial_0\psi_1,\psi_2)\wedge\bar\partial\omega_g^{n-1}}{\nu}\\ &= \frac{1}{n}\int_M h_0(\mathrm{tr}_g\bar\partial\partial_0\psi_1,\psi_2)\frac{\omega_g^n}{\nu}\\ &\quad -\int_M \frac{h_0(\partial_0\psi_1,\psi_2)\wedge\bar\partial\omega_g^{n-1}}{\nu}\end{aligned}$$

□

Proposition 21. *Assume E is a $\mathbb{C}$-simple complex flat vector bundle over M a compact special affine manifold. Suppose there is an $m\in\mathbb{R}$ so that $m_\epsilon\le m$ for all $\epsilon\in(\epsilon_0,1]$. Then for all $p>1$ and $\epsilon\in(\epsilon_0,1]$,*

$$\|\phi_\epsilon\|_{L^p_2}\le C(m)(1+\|f_\epsilon\|_{L^p_2}),$$

where $C(m)$ may depend on p as well as m and the initial data.

Proof. The variation $\phi=\phi_\epsilon$ satisfies

$$\begin{aligned}0 &= \frac{\delta}{\delta f}\hat L(\epsilon,f)(\phi)+f\log f\\ &= \phi[K_0-\gamma I_E+\epsilon\log f+\mathrm{tr}_g\bar\partial(f^{-1}\partial_0 f)]\\ &\quad -f\,\mathrm{tr}_g\bar\partial(f^{-1}\phi f^{-1}\partial_0 f)+f\,\mathrm{tr}_g\bar\partial(f^{-1}\partial_0\phi)\\ &\quad +f\log f+\epsilon f\left(\frac{\delta}{\delta f}\log f\right)(\phi).\end{aligned}$$

One computes then that

$$\begin{aligned}\mathrm{tr}_g(\bar\partial\partial_0\phi) &= -\phi(K_0-\gamma I_E+\epsilon\log f)-\mathrm{tr}_g(\bar\partial f\wedge f^{-1}\phi f^{-1}\partial_0 f)\\ &\quad +\mathrm{tr}_g(\bar\partial f\wedge f^{-1}\partial_0\phi)+\mathrm{tr}_g(\bar\partial\phi\wedge f^{-1}\partial_0 f)\\ &\quad -f\log f-\epsilon f\left(\frac{\delta}{\delta f}\log f\right)(\phi)\end{aligned}$$

Then Lemma 20 above shows that for the operator $\Lambda = n\,\partial_0^*\partial_0 + I_E$

$$\begin{aligned}\Lambda\phi &= -\phi[K_0 - (\gamma+1)I_E + \epsilon\log f] - \mathrm{tr}_g(\bar\partial f \wedge f^{-1}\phi f^{-1}\partial_0 f)\\ &\quad + \mathrm{tr}_g(\bar\partial f\wedge f^{-1}\partial_0\phi) + \mathrm{tr}_g(\bar\partial\phi\wedge f^{-1}\partial_0 f)\\ &\quad - f\log f - \epsilon f\left(\frac{\delta}{\delta f}\log f\right)(\phi) - n\,\frac{\partial_0\phi\wedge\bar\partial\omega_g^{n-1}}{\omega_g^n}.\end{aligned} \tag{19}$$

The operator $\Lambda\colon L^p_2(\mathrm{End}\,E)\to L^p(\mathrm{End}\,E)$ is elliptic, self-adjoint, and is continuously invertible, since $\partial_0^*\partial_0$ has nonnegative spectrum. Therefore, there is a C satisfying

$$\|\phi\|_{L^p_2} \le C\|\Lambda\phi\|_{L^p},$$

where as usual C depends only on the initial data and p.

So we consider the L^p norms of the 7 terms on the right-hand side of (19): The first term is bounded by $C(m)$ by Proposition 16, and the fifth is also bounded by $C(m)$. Proposition 16 and Hölder's inequality shows the second term is bounded by $C(m)\|f\|^2_{L^{2p}_1}$. The third and fourth terms are both bounded by $C(m)\|f\|_{L^{2p}_1}\|\phi\|_{L^{2p}_1}$. A local computation shows the sixth term is bounded by $C(m)$, and the last term is clearly bounded by $C\|\phi\|_{L^{2p}_1}$. So, altogether,

$$\|\phi\|_{L^p_2} \le C(m)(1+\|\phi\|_{L^{2p}_1} + \|\phi\|_{L^{2p}_1}\|f\|_{L^{2p}_1} + \|f\|^2_{L^{2p}_1}).$$

An interpolation inequality of Aubin [2, Theorem 3.69] states that

$$\|\psi\|_{L^{2p}_1} \le C\|\psi\|^{\frac12}_{L^\infty}\|\psi\|^{\frac12}_{L^p_2} + \|\psi\|_{L^{2p}}.$$

Since both $\|f\|_{L^\infty}, \|\phi\|_{L^\infty} \le C(m)$, a simple computation allows us to prove the proposition. □

Corollary 22. *Assume there is a smooth family of solutions f_ϵ to $L_\epsilon(f_\epsilon)=0$, and that there is a uniform m so that $m_\epsilon \le m$ for all $\epsilon\in(\epsilon_0,1]$. Then for all $\epsilon\in(\epsilon_0,1]$, $\|f_\epsilon\|_{L^p_2}\le C(m)$, where $C(m)$ does not depend on ϵ.*

Proof. Since $\phi_\epsilon = \frac{d}{d\epsilon}f_\epsilon$,

$$\frac{d}{d\epsilon}\|f_\epsilon\|_{L^p_2} \ge -\|\phi_\epsilon\|_{L^p_2} \ge -C(m)(1+\|f_\epsilon\|_{L^p_2}).$$

Then simply integrate this ordinary differential inequality. □

Proposition 23. *Assume E is a $\mathbb{C}$-simple flat complex vector bundle over M a compact special affine manifold. Then $J=(0,1]$. Moreover, if $\|f_\epsilon\|_{L^2}$ is bounded independently of $\epsilon\in(0,1]$, then there exists a smooth solution f_0 to the Hermitian-Einstein equation $L_0(f_0)=0$.*

Proof. The first statement will follow if we can show J is closed. In particular, all we need to show is that if $J = (\epsilon_0, 1]$ for $\epsilon_0 > 0$, then there is a smooth solution f_{ϵ_0} to $L_{\epsilon_0}(f_{\epsilon_0}) = 0$. Corollaries 18 and 22 and then shows there is a constant C satisfying $\|f_\epsilon\|_{L^p_2} \le C$ for all $\epsilon \in (\epsilon_0, 1]$. We will use this uniform estimate below to show the existence of f_{ϵ_0}.

Under the hypotheses of the second statement of the proposition, on the other hand, Corollaries 19 and 22 together show that there is a C so that for all $\epsilon \in (0, 1]$, $\|f_\epsilon\|_{L^p_2} \le C$.

Therefore, to prove the whole proposition, we may assume that for $\epsilon_0 \in [0, 1)$, there is a constant C and a smooth family of solutions f_ϵ of $L_\epsilon(f_\epsilon) = 0$ exists and satisfies $\|f_\epsilon\|_{L^p_2} \le C$. We will find a sequence $\epsilon_i \to \epsilon_0^+$ so that $f_{\epsilon_0} = \lim f_{\epsilon_i}$ is the solution we require.

Choose $p > n$. In this case, L^p_1 maps compactly into C^0, and so $\log : L^p_1(\operatorname{End} E) \to L^p_1(\operatorname{End} E)$ is continuous and the product of two functions in L^p_1 is also in L^p_1. (See e.g. [18].)

The uniform L^p_2 bound implies there is a sequence $\epsilon_i \to \epsilon_0$ so that $f_{\epsilon_i} \to f_{\epsilon_0}$ converges weakly in L^p_2, and strongly in L^p_1 and C^0. Then compute, in the sense of distributions, for α a smooth section of $\operatorname{End}(E)$,

$$\begin{aligned} \langle L_{\epsilon_0}(f_{\epsilon_0}), \alpha\rangle_{\operatorname{End}(E)} &= \langle L_{\epsilon_0}(f_{\epsilon_0}) - L_{\epsilon_i}(f_{\epsilon_i})\rangle_{\operatorname{End}(E)} \\ &= \int_M h_0(\operatorname{tr}_g[\bar\partial(f_{\epsilon_0}^{-1}\partial_0 f_{\epsilon_0} - f_{\epsilon_i}^{-1}\partial_0 f_{\epsilon_i})], \alpha)\,\frac{\omega_g^n}{\nu} \\ &\quad + \int_M h_0(\epsilon_0 \log f_{\epsilon_0} - \epsilon_i \log f_{\epsilon_i}, \alpha)\,\frac{\omega_g^n}{\nu} \end{aligned}$$

The second term goes to zero as $\epsilon_i \to \epsilon_0$ since $f_{\epsilon_i} \to f_{\epsilon_0}$ in C^0. Using Proposition 3, the first term can be written as

$$\begin{aligned} &n\int_M \frac{h_0(f_{\epsilon_0}^{-1}\partial_0 f_{\epsilon_0} - f_{\epsilon_i}^{-1}\partial_0 f_{\epsilon_i}, \partial_0\alpha) \wedge \omega_g^{n-1}}{\nu} \\ &+ n\int_M \frac{h_0(f_{\epsilon_0}^{-1}\partial_0 f_{\epsilon_0} - f_{\epsilon_i}^{-1}\partial_0 f_{\epsilon_i}, \alpha) \wedge \bar\partial\omega_g^{n-1}}{\nu}. \end{aligned}$$

Both these terms converge to 0 since $f_{\epsilon_i}^{-1}\partial_0 f_{\epsilon_i} \to f_{\epsilon_0}^{-1}\partial_0 f_{\epsilon_0}$ in L^p. Therefore, $L_{\epsilon_0}(f_{\epsilon_0}) = 0$ in the sense of distributions.

Now we can compute in much the same way, for $f_{\epsilon_0} \in L^p_2$, $\operatorname{tr}_g\bar\partial\partial_0 f_{\epsilon_0} \in L^p_1$. Therefore, $f_{\epsilon_0} \in L^p_3$, and we can bootstrap further to show that f_{ϵ_0} is smooth and is a classical solution to $L_{\epsilon_0}(f_{\epsilon_0}) = 0$. □

10. Construction of a destabilizing subbundle

In this section, we will construct a destabilizing flat subbundle of E if $\limsup_\epsilon \|f_\epsilon\|_{L^2} = \infty$. For a sequence $\epsilon_i \to 0$, we will rescale by the reciprocal ρ_i of the largest eigenvalue of f_{ϵ_i}. Then we will show that the limit

$$\lim_{\sigma\to 0}\lim_{i\to\infty}(\rho_i f_{\epsilon_i})^\sigma$$

exists and all of its eigenvalues are 0 or 1. A projection to the destabilizing subbundle will be given by I_E minus this limit.

Proposition 24. *If $\epsilon > 0$, $0 < \sigma \le 1$, and f satisfies $L_\epsilon(f) = 0$, then*

$$-\frac{1}{\sigma}\,\mathrm{tr}_g\partial\bar\partial(\mathrm{tr} f^\sigma) + \epsilon\, h_0(\log f, f^\sigma) + |f^{-\frac{\sigma}{2}}\partial_0(f^\sigma)|^2 \le -h_0(K_0 - \gamma I_E, f^\sigma).$$

Proof. This is a local computation, for which we refer to [18, Lemma 3.4.4]. □

To rescale f_ϵ properly, consider the largest eigenvalue $\lambda(\epsilon, x)$ of $\log f_\epsilon(x)$ for $x \in M$, and define

$$M_\epsilon = \max_{x\in M}\lambda(\epsilon, x), \qquad \rho_\epsilon = e^{-M_\epsilon}.$$

Then since $\det f_\epsilon = 1$, $\rho_\epsilon \le 1$ and we have the following straightforward lemma:

Lemma 25. *Assume* $\limsup_{\epsilon\to 0}\|f_\epsilon\|_{L^2} = \infty$. *Then*

(1) $\rho_\epsilon f_\epsilon \le I_E$.
(2) *For each $x \in M$, there is an eigenvalue of $\rho_\epsilon f_\epsilon$ less than or equal to ρ_ϵ.*
(3) $\max_M \rho_\epsilon |f_\epsilon| \ge 1$.
(4) *There is a sequence $\epsilon_i \to 0$ so that $\rho_{\epsilon_i} \to 0$.*

Proposition 26. *There is a subsequence $\epsilon_i \to 0$ so that $\rho_{\epsilon_i} \to 0$ and so that $f_i = \rho_{\epsilon_i} f_{\epsilon_i}$ satisfies*

(1) *f_i converges weakly in L^2_1 to an $f_\infty \ne 0$.*
(2) *As $\sigma \to 0$, f^σ_∞ converges weakly in L^2_1 to f^0_∞.*

Proof. First of all, note that since each f^σ_ϵ is positive-definite and self-adjoint with respect to h_0,

$$|f^\sigma_\epsilon| \le \mathrm{tr}\, f^\sigma_\epsilon \le \sqrt{r}\,|f^\sigma_\epsilon|. \tag{20}$$

Let $\sigma \in (0, 1]$. Then Proposition 24, Corollary 18, and (20) show

$$\begin{aligned}\mathrm{tr}_g\partial\bar\partial(\mathrm{tr}\, f^\sigma_\epsilon) &\ge \epsilon\, h_0(\log f_\epsilon, f^\sigma_\epsilon) + h_0(K_0 - \gamma I_E, f^\sigma_\epsilon)\\ &\ge -(\epsilon m_\epsilon + C)|f^\sigma_\epsilon|\\ &\ge -C|f^\sigma_\epsilon| \ge -C\,\mathrm{tr}\, f^\sigma_\epsilon,\end{aligned}$$

where, as usual, C is a (changing) constant depending only on the initial data. Now Proposition 15, Lemma 25 and (20) show that

$$1 \le \max_M \rho^\sigma_\epsilon |f^\sigma_\epsilon| \le \max_M \rho^\sigma_\epsilon \mathrm{tr}\, f^\sigma_\epsilon \le C\rho^\sigma_\epsilon \|\mathrm{tr}\, f^\sigma_\epsilon\|_{L^1} \le C\|\rho^\sigma_\epsilon f^\sigma_\epsilon\|_{L^2}. \tag{21}$$

On the other hand, Lemma 25 shows

$$\|\rho_\epsilon^\sigma f_\epsilon^\sigma\|_{L^2} \leq \|I_E\|_{L^2} = C,$$

and so it remains to estimate $\|\partial_0(f_i^\sigma)\|_{L^2}$ to get uniform bounds on $\|f_i^\sigma\|_{L^2_1}$.

Compute for $\epsilon = \epsilon_i$,

$$\begin{aligned}
\|\partial_0 f_i^\sigma\|_{L^2}^2 &= \int_M |\partial_0(\rho_\epsilon^\sigma f_\epsilon^\sigma)|^2 \,\frac{\omega_g^n}{\nu} \\
&\leq \int_M |(\rho_\epsilon f_\epsilon)^{-\frac{\sigma}{2}} \partial_0(\rho_\epsilon^\sigma f_\epsilon^\sigma)|^2 \,\frac{\omega_g^n}{\nu} \\
&\leq \rho_\epsilon^\sigma \int_M \frac{1}{\sigma} \mathrm{tr}_g \partial\bar\partial(\mathrm{tr} f_\epsilon^\sigma)\frac{\omega_g^n}{\nu} - \rho_\epsilon^\sigma \int_M h_0(\epsilon \log f_\epsilon + K_0 - \gamma I_E, f_\epsilon^\sigma)\,\frac{\omega_g^n}{\nu} \\
&= \frac{\rho_\epsilon^\sigma n}{\sigma} \int_M \frac{\partial\bar\partial(\mathrm{tr} f_\epsilon^\sigma)\wedge \omega_g^{n-1}}{\nu} - \int_M h_0(\epsilon \log f_\epsilon + K_0 - \gamma I_E, \rho_\epsilon^\sigma f_\epsilon^\sigma)\,\frac{\omega_g^n}{\nu} \\
&= -\int_M h_0(\epsilon \log f_\epsilon + K_0 - \gamma I_E, \rho_\epsilon^\sigma f_\epsilon^\sigma)\,\frac{\omega_g^n}{\nu} \\
&\leq C \max_M (\rho_\epsilon f_\epsilon)^\sigma \leq C,
\end{aligned}$$

where we have used Lemma 25 to show $(\rho_\epsilon f_\epsilon)^{-\frac{\sigma}{2}} \geq I_E$ to derive the second line from the first; Proposition 24 for the third line; Proposition 3, Stokes' Theorem, and the fact that g is affine Gauduchon to get the fifth line; and finally Corollary 18 and Lemma 25 to derive the sixth line. Note the final bound C is independent of σ and ϵ.

For $\sigma = 1$, therefore, we have uniform L^2_1 bounds on f_i, and so there is an L^2_1-weakly-convergent subsequence which we may assume converges in L^2 and almost everywhere on M. For simplicity, we still call this subsequence f_i. The bound (21) shows that $f_\infty = \lim f_i$ is not zero in L^2.

The almost everywhere convergence of $f_i \to f_\infty$ shows that f_∞ is h_0-adjoint and positive semidefinite almost everywhere. Lemma 25 shows that each eigenvalue of f_∞ is in $[0, 1]$. Therefore, by considering a (measurable) frame which diagonalizes f_∞ at almost every point, it is clear that f_∞^σ converges to a limit f_∞^0 pointwise almost everywhere as $\sigma \to 0$.

Moreover, the uniform bounds on $\|f_i^\sigma\|_{L^2_1}$ for all $\sigma \in (0, 1]$ show that $\|f_\infty^\sigma\|_{L^2_1}$ is also uniformly bounded independent of σ, and so for each sequence $\sigma_j \to 0$, there is a subsequence σ_{j_k} so that $f_\infty^{\sigma_{j_k}}$ converges weakly in L^2_1, strongly in L^2 and pointwise almost everywhere to f_∞^0. Thus $f_\infty^\sigma \to f_\infty^0$ weakly in L^2_1 as $\sigma \to 0$. □

Now let $\pi = I_E - f^0_\infty$.

Proposition 27. *The endomorphism $\pi = I_E - f^0_\infty$ is an h_0-orthogonal projection onto a flat subbundle of E. In other words, it satisfies $\pi^2 = \pi$, $\pi^* = \pi$ and $(I_E - \pi)\bar\partial\pi = 0$ in L^1. Moreover, π is a smooth endomorphism of E. So the locally constant subbundle $F = \pi(E)$ is smooth.*

Proof. First we show that $\pi = \pi^*$, $\pi = \pi^2$, and $(1-\pi)\bar\partial\pi = 0$ in L^1 only. Then we will finish the proof with a discussion of regularity.

To show $\pi = \pi^*$ almost everywhere, recall f^0_∞ is a pointwise almost-everywhere limit of f^σ_∞, and f_∞ is a pointwise almost-everywhere limit of f_i, which satisfies $f_i = f_i^*$.

To show $\pi^2 = \pi$ in L^1, use Proposition 26 to compute

$$\pi^2 = \lim_{\sigma\to 0}(I_E - f^\sigma_\infty)^2 = I_E - 2\lim_{\sigma\to 0}(f^\sigma_\infty + f^{2\sigma}_\infty) = 1 - 2f^0_\infty + f^0_\infty = \pi.$$

To show $(1-\pi)\bar\partial\pi = 0$ in L^1, compute since $\pi = \pi^* = \pi^2$ that

$$|(I_E - \pi)\bar\partial\pi| = |\bar\partial(I_E-\pi)\pi| = |[\bar\partial(I_E-\pi)\pi]^*| = |\pi\partial_0(I_E-\pi)|.$$

(Here * represents the adjoint with respect to h_0 only, and not with respect to any Hodge-type star on the affine Dolbeault complex $\mathcal{A}^{p,q}(\operatorname{End} E)$.) So we will show that

$$\|\pi\partial_0(I_E-\pi)\|_{L^2} = 0.$$

Since the eigenvalues of f_i are between 0 and 1, a local computation (see e.g. [18, p. 87]) implies that

$$0 \le \frac{s+\frac{\sigma}{2}}{s}(I_E - f_i^s) \le f_i^{-\frac{\sigma}{2}}$$

for $0 \le s \le \frac{\sigma}{2}$. Then, as above, Proposition 24 shows that

$$\begin{aligned}
\int_M |(I_E - f_i^s)\partial_0(f_i^\sigma)|^2\frac{\omega_g^n}{\nu} &\le \left(\frac{s}{s+\frac{\sigma}{2}}\right)^2\int_M |f_i^{-\frac{\sigma}{2}}\partial_0(f_i^\sigma)|^2\frac{\omega_g^n}{\nu}\\
&\le \left(\frac{s}{s+\frac{\sigma}{2}}\right)^2\int_M |\epsilon_i\log f_i + K_0 - \gamma I_E||f_i|^\sigma\frac{\omega_g^n}{\nu}\\
&\le \left(\frac{s}{s+\frac{\sigma}{2}}\right)^2 C.
\end{aligned}$$

Since $\{(I_E - f_i^s)\partial_0(f_i^\sigma)\}_{i=1}^\infty$ is a bounded sequence in L^2, weak compactness in L^2 allows us to take $i\to\infty$ to find

$$\int_M |(I_E - f^s_\infty)\partial_0(f^\sigma_\infty)|^2\frac{\omega_g^n}{\nu} \le \left(\frac{s}{s+\frac{\sigma}{2}}\right)^2 C.$$

Now we let $s \to 0$ first so that $I_E - f^s_\infty \to I_E - f^0_\infty = \pi$ strongly in L^2 as $s \to 0$ by the uniform L^2_1 bounds. So

$$\int_M |\pi \partial_0 (f^\sigma_\infty)| \frac{\omega_g^n}{\nu} = 0.$$

By definition, $\lim_{\sigma \to 0} \partial_0 f^\sigma_\infty$ converges weakly in L^2 to $\partial_0(I_E - \pi)$, and so $\int_M |\pi \partial_0 (I_E - \pi)| \frac{\omega_g^n}{\nu} = 0$.

It remains to show that $\pi = \pi^2 = \pi^*$ and $\pi \bar\partial (I_E - \pi) = 0$ in L^1 implies that π is smooth. The regularity of $F = \pi(E)$ is a local issue, and so we restrict to a local coordinate chart and a locally constant frame. By an argument of Popovici [19, Lemma 0.3.3], we can assume h_0 is the standard flat metric with regards to the locally constant frame.

In terms of the standard flat metric, in order to show that $F = \pi(E)$ is a smooth flat vector bundle, it suffices to show that

$$\bar\partial \pi = 0 \qquad \Longleftrightarrow \qquad \nabla \pi = 0.$$

At each $x \in M$, $\pi(x)$ can be considered as a map from $\mathbb{C}^r$ to $\mathbb{C}^r$ of some rank k. The conditions π satisfies are then

$$\pi^2 = \pi, \qquad \pi^* = \pi, \qquad (I_E - \pi)\bar\partial \pi = 0,$$

for * the conjugate transpose. Now π is L^2_1 when restricted to almost every coordinate line segment, with variable t on the segment. Then the last condition on π becomes

$$(I - \pi)\frac{d\pi}{dt} = (I - \pi)\dot\pi = 0.$$

The adjoint of this equation is then

$$0 = (\dot\pi)^*(I - \pi)^* = \dot\pi (I - \pi).$$

Differentiating $\pi^2 = \pi$ and applying $\dot\pi = \pi\dot\pi$, we also have

$$\dot\pi \pi = 0.$$

Adding these two equations shows that

$$\dot\pi = (I - \pi)\dot\pi + \pi\dot\pi = 0$$

in the sense of distributions. So π is constant along almost every coordinate line segment. Then it is easy to see that π is constant almost everywhere, and thus is equal to a constant matrix in the sense of distributions.

We should remark that this simple proof works because d/dt is a real operator. More properly, on an affine manifold, $\bar\partial$ is a real operator: We may ignore our convention (2), and instead map $\bar\partial$ to the real operator $\frac{1}{2}\nabla$ via the a natural map from $\mathcal{A}^{0,1}(\operatorname{End} E) \to \Lambda^1(\operatorname{End} E)$ induced by $dz^i \mapsto dx^i$. So $\pi^* = \pi$ implies $\dot\pi^* = \dot\pi$. This fails in the case of complex manifolds, and the proof to show that the image of π is a coherent analytic subsheaf is quite a bit more involved (Uhlenbeck-Yau [24, 25]), although see the simplification by Popovici [19]. $\square$

Proposition 28. *The flat subbundle $F = \pi(E) \subset E$ is a proper subbundle. In other words,*

$$0 < \operatorname{rank} F < \operatorname{rank} E.$$

Proof. First of all, note that $\operatorname{rank} F$ is a constant over M, since it is equal to the rank of π as an endomorphism, and π is locally constant.

Now $f^0_\infty = \lim_{\sigma\to 0} f^\sigma_\infty$ is not identically zero since $f_\infty \neq 0$ (Proposition 26), and the eigenvalues of f^σ_∞ are nonnegative and nondecreasing as $\sigma \to 0$. So $\pi = I_E - f^0_\infty$ is not identically I_E. Since π is a projection, $\operatorname{rank}\pi < \operatorname{rank} E$.

On the other hand, Lemma 25 (there is everywhere on M an eigenvalue of f_i which is bounded by $\rho_i \to 0$) shows that f_∞ has a nontrivial kernel at almost every point. Therefore, f^0_∞ does as well, and $\pi = I_E - f^0_\infty$ cannot be identically 0. So $\operatorname{rank}\pi > 0$. $\square$

Proposition 29. *The flat subbundle $F = \pi(E)$ is a destabilizing subbundle of E. In other words,*

$$\frac{\deg_g E}{\operatorname{rank} E} = \mu_g E \leq \mu_g F = \frac{\deg_g F}{\operatorname{rank} F}.$$

Proof. Recall

$$\mu_g E = \frac{1}{r}\int_M \frac{c_1(E,h)\wedge\omega_g^{n-1}}{\nu} = \frac{1}{nr}\int_M \operatorname{tr} K_0\,\frac{\omega_g^n}{\nu},$$

and for $s = \operatorname{rank} F$ and K_F the extended mean curvature of the extended Hermitian connection on F with respect to the Hermitian metric $h_0|_F$ the restriction of h_0 to F.

$$\mu_g F = \frac{1}{s}\int_M \frac{c_1(F,h_0|_F)\wedge\omega_g^{n-1}}{\nu} = \frac{1}{ns}\int_M \operatorname{tr} K_F\,\frac{\omega_g^n}{\nu}.$$

The Chern-Weil formula (see e.g. Kobayashi [13]) shows that $\operatorname{tr} K_F = \operatorname{tr}(K_0\pi) - |\pi^\perp\partial_0\pi|^2$ for $\pi^\perp\partial_0\pi$ the second fundamental form of the subbundle $F\subset E$. Now

$$\pi^\perp\partial_0\pi = (I_E - \pi)\partial_0\pi = \partial_0\pi - \pi\partial_0\pi = \partial_0\pi.$$

If we define $K^0 = K_0 - \gamma I_E$, then $\operatorname{tr} K^0 = 0$ and

$$\mu_g F = \frac{1}{ns}\int_M [\operatorname{tr}(K^0\pi) - |\partial_0\pi|^2]\,\frac{\omega_g^n}{\nu} + \frac{\gamma}{n}\int_M \frac{\omega_g^n}{\nu},$$

while (8) shows $\mu_g E = \frac{\gamma}{n}\int_M \frac{\omega_g^n}{\nu}$. Therefore, in order to show $\mu_g F \geq \mu_g E$, we need to show

$$\int_M \operatorname{tr}(K^0\pi)\frac{\omega_g^n}{\nu} \geq \int_M |\partial_0\pi|^2\frac{\omega_g^n}{\nu}. \tag{22}$$

Since $\pi = \lim_{\sigma\to 0}\lim_{i\to\infty}(I_E - f_i^\sigma)$ strongly in L^2 and $\operatorname{tr} K^0 = 0$,

$$\int_M \operatorname{tr}(K^0\pi)\frac{\omega_g^n}{\nu} = -\lim_{\sigma\to 0}\lim_{i\to\infty}\int_M \operatorname{tr}(K^0 f_i^\sigma)\frac{\omega_g^n}{\nu}.$$

Compute, using equation (14),

$$\begin{aligned}
-\int_M \operatorname{tr}(K^0 f_i^\sigma)\frac{\omega_g^n}{\nu} &= \int_M \epsilon_i \operatorname{tr}(\log f_{\epsilon_i}\cdot f_i^\sigma)\frac{\omega_g^n}{\nu} \\
&\quad + \int_M \operatorname{tr}\{[\operatorname{tr}_g\bar\partial(f_i^{-1}\partial_0 f_i)]f_i^\sigma\}\frac{\omega_g^n}{\nu} \\
&\geq \int_M \operatorname{tr}\{[\operatorname{tr}_g\bar\partial(f_i^{-1}\partial_0 f_i)]f_i^\sigma\}\frac{\omega_g^n}{\nu} \\
&= n\int_M \frac{\operatorname{tr}\{[\bar\partial(f_i^{-1}\partial_0 f_i)]f_i^\sigma\}\wedge\omega_g^{n-1}}{\nu} \\
&= n\int_M \frac{\operatorname{tr}[(f_i^{-1}\partial_0 f_i)\wedge\bar\partial(f_i^\sigma)]\wedge\omega_g^{n-1}}{\nu} \\
&\quad + n\int_M \frac{\operatorname{tr}[(f_i^{-1}\partial_0 f_i)f_i^\sigma]\wedge\bar\partial\omega_g^{n-1}}{\nu},
\end{aligned}$$

where the inequality follows from a local calculation as in [18, p. 89] and the last equality follows from Proposition 3 and integration by parts. Now a local computation shows that the last integral above satisfies

$$\int_M \frac{\operatorname{tr}[(f_i^{-1}\partial_0 f_i)f_i^\sigma]\wedge\bar\partial\omega_g^{n-1}}{\nu} = \frac{1}{\sigma}\int_M \frac{\partial[\operatorname{tr}(f_i^\sigma)]\wedge\bar\partial\omega_g^{n-1}}{\nu} = 0$$

by integration by parts since g is affine Gauduchon. On the other hand, the other term

$$\begin{aligned} n\int_M \frac{\mathrm{tr}[(f_i^{-1}\partial_0 f_i)\wedge\bar\partial(f_i^\sigma)]\wedge\omega_g^{n-1}}{\nu} &= \int_M \mathrm{tr}\,\mathrm{tr}_g[(f_i^{-1}\partial_0 f_i)\wedge\bar\partial(f_i^\sigma)]\frac{\omega_g^n}{\nu} \\ &= \int_M \mathrm{tr}_g\, h_0(f_i^{-1}\partial_0 f_i, \partial_0(f_i^\sigma))\frac{\omega_g^n}{\nu} \\ &\geq \int_M |f_i^{-\frac{\sigma}{2}}\partial_0(f_i^\sigma)|^2\frac{\omega_g^n}{\nu} \\ &\geq \|\partial_0(f_i^\sigma)\|_{L^2}^2 \\ &= \|\partial_0(I_E - f_i^\sigma)\|_{L^2}^2. \end{aligned}$$

Here, the second line follows from the first since $h_0(A,B) = \mathrm{tr}(AB^*)$ for B^* the h_0-adjoint of B, the third line follows by a local computation [18, Lemma 3.4.4.i], and the fourth line follows since $f_i \leq I_E$.

Therefore,

$$-\int_M \mathrm{tr}(K^0 f_i^\sigma)\frac{\omega_g^n}{\nu} \geq \|\partial_0(I_E - f_i^\sigma)\|_{L^2}^2,$$

and since $\partial_0\pi$ is the weak L^2 limit of $\partial_0(I_E - f_i^\sigma)$,

$$\lim_{\sigma\to 0}\lim_{i\to\infty}\|\partial_0(I_E - f_i^\sigma)\|_{L^2}^2 \geq \|\partial_0\pi\|_{L^2}^2.$$

This proves the proposition. □

This proposition completes the proof of Theorem 1.

11. Simple bundles

Some of this section is a simplified version of Kobayashi [13, Section V.7].

Proposition 30. *Every $\mathbb{C}$-stable flat vector bundle E over a compact special affine manifold M is $\mathbb{C}$-simple.*

Proof. Consider a locally constant section f of $E^*\otimes E$, and let $a \in \mathbb{C}$ be an eigenvalue of E at a point $x \in M$. Then $f - aI_E$ is a locally constant endomorphism of E which has a 0 eigenvalue at x. Consider $H = (f - aI_E)(E)$. Thus $\mathrm{rank} H < \mathrm{rank} E$. We use the $\mathbb{C}$-stability to show $H = 0$. If $\mathrm{rank} H > 0$, then the stability of E implies that

$$\mu(H) < \mu(E).$$

But we can also identify H with the quotient bundle $E/\ker(f - aI_E)$, which implies

$$\mu(E) < \mu(H),$$

which provides a contradiction. Thus $H = 0$ and $f = aI_E$ for the constant $a \in \mathbb{C}$.

Thus the proposition follows from the following $\square$

Proposition 31. *If E is a $\mathbb{C}$-stable flat vector bundle over a compact special affine manifold M, then any flat quotient vector bundle H over E satisfies $\mu(E) > \mu(H)$.*

Proof. If

$$0 \to F \to E \to H \to 0$$

is an exact sequence of flat vector bundles on M, then

$$\deg F + \deg H = \deg E \tag{23}$$

The proof of (23) is to compute the affine first Chern form.

In terms of a locally constant frame $s_1, \dots, s_r$ of E, and for $h_{\alpha\bar\beta} = h(s_\alpha, s_\beta)$ as above, the first Chern form is

$$c_1(E, h) = -\partial\bar\partial \log \det h_{\alpha\bar\beta}. \tag{24}$$

We will show that there are natural frames and metrics so that $c_1(E) = c_1(F) + c_1(H)$.

On each sufficiently small open set $U \subset M$, there is a locally constant frame $\{s_1, \dots, s_r\}$ so that $\{s_1, \dots, s_{r'}\}$ is a locally constant frame of the subbundle F (for $r' \le r$ the rank of F). Then the equivalence classes $\{[s_{r'+1}], \dots [s_r]\}$ form a locally constant frame of the quotient bundle H (here, at $x \in U$, $[s(x)] = s(x) + F_x \in E_x/F_x = H_x$).

We assume E admits a Hermitian metric h. Then $h|_F$ is a Hermitian metric on F. Now there is an orthonormal frame $\{t_1, \dots, t_r\}$ of E so that $t_1, \dots, t_{r'}$ are sections of F. Then the change-of-frame matrix $A = (A_\alpha^\beta)$ satisfying $t_\alpha = A_\alpha^\beta s_\beta$ is block-triangular of the form

$$A = \begin{pmatrix} P & * \\ 0 & Q \end{pmatrix}, \tag{25}$$

where P is the change-of-frame matrix on F taking $\{s_1, \dots, s_{r'}\}$ to $\{t_1, \dots, t_{r'}\}$. The metric h allows us to identify the quotient bundle H with the orthogonal complement $F^\perp$ of F in E by orthogonal projection. Under this identification, the matrix Q is the change-of-frame matrix on $F^\perp$ taking $\{[s_{r'+1}], \dots, [s_r]\}$ to $\{t_{r'+1}, \dots, t_r\}$. Note (25) shows $\det A = (\det P)(\det Q)$.

Now note that the metric $h = (h_{\alpha\bar\beta})$ can be recovered from a change of frame matrix A by $h = (A\bar{A}^\perp)^{-1}$—i.e., $A_\alpha^\gamma h_{\gamma\epsilon} \bar{A}_\beta^\epsilon = \delta_{\alpha\beta}$ for the Kronecker $\delta_{\alpha\beta}$. Then the formulas (24) and (25) show that $c_1(E) = c_1(F) + c_1(H)$.

So the degree addition formula (23) follows from the definition (4). Now

$$\mu_g(F) < \mu_g(E) \qquad \Longleftrightarrow \qquad \mu_g(H) > \mu_g(E),$$

which proves the proposition. □

Finally, we consider the case of real flat vector bundles. Now let E be a real flat vector bundle over a compact special affine manifold M equipped with an affine Gauduchon metric g. Such a vector bundle E is said to be $\mathbb{R}$-stable if every real flat subbundle F of E satisfies

$$0 < \operatorname{rank} F < \operatorname{rank} E \qquad \Longrightarrow \qquad \mu_g(F) < \mu_g(E).$$

It is obvious that the $\mathbb{C}$-stability of $E \otimes_{\mathbb{R}} \mathbb{C}$ implies the $\mathbb{R}$-stability of E, but the converse may not be true.

Proposition 32. *Let E be an $\mathbb{R}$-stable flat real vector bundle over M a compact special affine manifold. As a complex flat vector bundle, $E \otimes_{\mathbb{R}} \mathbb{C}$ satisfies one of the following:*

- *$E \otimes_{\mathbb{R}} \mathbb{C}$ is $\mathbb{C}$-simple.*
- *$E \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \bar{V}$, where V is a $\mathbb{C}$-stable flat complex vector subbundle of $E \otimes_{\mathbb{R}} \mathbb{C}$ and $\bar{V}$ is its complex conjugate as a subbundle of $E \otimes_{\mathbb{R}} \mathbb{C}$.*

Proof. Case 1: Every real locally constant section of $\operatorname{End} E$ has only real eigenvalues at every point $x \in M$. In this case, let f be a real locally constant section of $\operatorname{End} E$, and let $a \in \mathbb{R}$ be an eigenvalue of f at a point $x \in M$. Then $f - aI_E$ is a real section of $\operatorname{End} E$ and, following the proof of Proposition 30 above, $f - aI_E$ must be identically 0, since E is $\mathbb{R}$-stable. So $f = aI_E$. The same is true for a complex locally constant section of $\operatorname{End} E$ by considering real and imaginary parts. Thus $E \otimes_{\mathbb{R}} \mathbb{C}$ is $\mathbb{C}$-simple in this case.

Case 2: There is a real locally constant section f of $\operatorname{End} E$ with an eigenvalue $a \notin \mathbb{R}$ at a point $x \in M$. Then $g = (f - aI) \circ (f - \bar{a}I)$ is a real section of $\operatorname{End}(E)$. Again, as in the proof of Proposition 30, g must be identically 0. So we have the following splitting into eigenbundles

$$E \otimes_{\mathbb{R}} \mathbb{C} = E_a \oplus E_{\bar{a}} = E_a \oplus \overline{E_a}.$$

Now we show that E_a and $E_{\bar{a}}$ must each be $\mathbb{C}$-stable. Let F be a flat complex subbundle of E_a. Then it is easy to see that $F \oplus \bar{F}$ is a real subbundle of $E_a \oplus \overline{E_a} = E \otimes_{\mathbb{R}} \mathbb{C}$. The $\mathbb{C}$-stability of E_a follows from the observation that the slope $\mu(F) = \mu(F \oplus \bar{F})$ for any flat subbundle F of E_a.

This observation may be proved by noting that $\operatorname{rank}(F \oplus \bar{F}) = 2 \operatorname{rank} F$, and that the degree $\deg(F \oplus \bar{F}) = 2 \deg F$ also. The degree calculation can be verified by choosing a Hermitian metric h on F

and extending it to $F \oplus \bar{F}$ by setting

$$h(\xi, \bar{\eta}) = h(\bar{\xi}, \eta) = 0, \qquad h(\bar{\xi}, \bar{\eta}) = \overline{h(\xi, \eta)} \tag{26}$$

for ξ, η sections of F. □

Corollary 33. *Any $\mathbb{R}$-stable flat real vector bundle E over a compact special affine manifold M admits a real Hermitian-Einstein metric.*

Proof. If E is $\mathbb{C}$-stable, then we are done. If not, the previous proposition shows that $E \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \bar{V}$ for V a complex stable flat subbundle. Then V admits a Hermitian-Einstein metric. It extends to a real Hermitian-Einstein metric on $E \otimes_{\mathbb{R}} \mathbb{C}$ by using (26) above. □

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