# SURVEY ON AFFINE SPHERES

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## 1. Introduction

Affine spheres were introduced by Ţiţeica in [74, 75, 76], and studied later by Blaschke, Calabi, and Cheng-Yau, among others. These are hypersurfaces in affine $\mathbb{R}^{n+1}$ which are related to real Monge-Ampère equations, to projective structures on manifolds, and to the geometry of Calabi-Yau manifolds. In this survey article, we will outline the theory of affine spheres their relationships to these topics.

Affine differential geometry is the study of those differential properties of hypersurfaces of $\mathbb{R}^{n+1}$ which are invariant under all volume-preserving affine transformations. Affine differential geometry is largely traced to Ţiţeica's papers in 1907-09, although for curves in $\mathbb{R}^2$, one of the main invariants, the affine normal, was already introduced by Transon [71] in 1841. (See [1] for a historical account of the Ţiţeica's development of affine differential geometry.) Given a smooth hypersurface $H \subset \mathbb{R}^{n+1}$, the affine normal $\xi$ is an affine-invariant transverse vector field to $H$. Define the special affine group as

$$\mathbf{SA}(n+1,\mathbb{R}) = \{\Phi \colon x \mapsto Ax + b,\ \det A = 1\}.$$

The invariance property of the affine normal is then

$$\Phi_* \xi_H(x) = \xi_{\Phi(H)}(\Phi(x))$$

for any $x \in H$. An *improper affine sphere* is a hypersurface $H$ whose affine normals are all parallel, while a *proper affine sphere* is a hypersurface whose affine normal lines all meet in a point, the *center* of the affine sphere. By symmetry, a Euclidean sphere must be a proper affine sphere, and affine invariance then shows that all ellipsoids are affine spheres also. More generally, quadric hypersurfaces are the canonical examples of affine spheres.

We will mainly focus on the case of convex hypersurfaces in this survey, since in this case the natural invariant metric, the affine metric, is positive definite, and so we can exploit techniques of Riemannian geometry and elliptic PDEs. We assume for a convex hypersurface that the affine normal points to the convex side of the hypersurface. Improper affine sphere are then called *parabolic affine spheres*, and the primary example is an elliptic paraboloid. For convex hypersurfaces, there are naturally two types of proper affine spheres, depending on whether the affine normal points toward or away from the center. For an *elliptic affine sphere*, such as an ellipsoid, the affine normals point inward toward the center. *Hyperbolic affine spheres* have affine normals which point away from the center. One component of a hyperboloid of two sheets is the quadric example of a hyperbolic affine sphere.

Some 15 years Ţiţeica's papers, Blaschke's book [7] records much of the early development of affine geometry. Calabi's papers contain many advances in the theory of affine spheres and related subjects, and Cheng-Yau's resolution of the structure of hyperbolic affine spheres provides crucial analytic estimates related to Monge-Ampère equations [13, 15]. We refer the reader to the books of Nomizu-Sasaki [59] and Li-Simon-Zhao [46] for overviews of affine differential geometry. Nomizu-Sasaki [59] develop the theory in the modern notation of connections, while Li-Simon-Zhao [46] use Cartan's moving frame techniques and also provide many analytic details.

This survey article focuses on the relationship between the geometry of affine spheres to geometric structures on manifolds. To this end in Section 4, we describe the semi-linear PDE of Ţiţeica and Wang involving the cubic differential and developing map for affine spheres in $\mathbb{R}^3$. Then we outline the relationship, due to Blaschke and Calabi, of affine spheres to real Monge-Ampère equations and the basic duality results related related to the Legendre transform in Section 5. In Sections 7-8, we discuss Cheng-Yau's work on hyperbolic affine spheres and invariants of convex cones. In Section 9, we relate affine spheres to the geometry of affine manifolds, and the conjecture of Strominger-Yau-Zaslow, relating parabolic affine spheres to Calabi-Yau manifolds. In the last two sections, we discuss two generalizations of affine spheres: Affine maximal hypersurfaces are generalizations of parabolic affine spheres which have led to significant progress in the theory of fourth-order elliptic equations in the solution of Chern's conjecture for affine maximal surfaces in $\mathbb{R}^3$ by Trudinger-Wang [72]. We also briefly discuss the affine normal flow, which is the natural parabolic analog of the elliptic PDEs of affine spheres. The selection of topics reflects the author's interests, and there are many important subjects which lie outside the author's expertise. We largely do not treat nonconvex affine spheres, and we only mention a few of the recent results classifying affine hypersurfaces with extremal geometric conditions.

This article is dedicated to Prof. S.T. Yau, who introduced me to affine differential geometry, for his rich insight and kind encouragement over the years, on the occasion of his 59$^{\text{th}}$ birthday.

## 2. Affine structure equations

Blaschke used the invariance of the affine normal to derive other affine-invariant quantities, such as the affine metric and cubic form [7]. As in e.g. Nomizu-Sasaki [59], the invariance of the affine normal can be demonstrated by putting affine-invariant conditions on arbitrary transverse vector fields to hypersurfaces. So let $L$ be a smooth strictly convex hypersurface in $\mathbb{R}^{n+1}$ and let $\bar\xi$ be a transverse vector field. Then we have the following structure equations of Gauss and Weingarten:

$$\begin{aligned} D_X Y &= \bar\nabla_X Y + \bar h(X,Y)\bar\xi, \\ D_X\bar\xi &= -\bar S(X) + \bar\tau(X)\bar\xi, \end{aligned}$$

where $X, Y$ are tangent vector fields to $L$, $D$ is the standard affine connection on $\mathbb{R}^{n+1}$, and the equations are given by the splitting at each $x \in L$

$$T_x\mathbb{R}^{n+1} = T_x L + \langle\bar\xi\rangle$$

of the tangent space to $\mathbb{R}^{n+1}$ into the tangent space to $L$ and the span of $\bar\xi$. In this formulation, $\bar\nabla$ is a torsion-free connection on $L$, $\bar h$ is a symmetric tensor, $\bar S$ is an endomorphism of the tangent bundle $TL$ and $\bar\tau$ is a one-form.

The affine normal $\xi$ to a convex hypersurface $H$ is the unique transverse vector field satisfying

- $\xi$ points to the convex side of $L$ (this is equivalent to $h$ being positive-definite).
- $\xi$ is *equiaffine*, which means that $\tau = 0$.
- For $X_i$ a frame of the tangent bundle $TL$,

$$\det_{1\le i,j\le n} h(X_i, X_j) = \det(X_1,\dots,X_n,\xi)^2,$$

where the second determinant is that on $\mathbb{R}^{n+1}$. This approach is similar to that of [59].

**Proposition 1.** *The affine normal is well-defined on any smooth strictly convex hypersurface $H \subset \mathbb{R}^{n+1}$.*

*Proof.* Consider any transversal vector field $\bar\xi$ which points to the convex side of $L$. If we set

$$\xi = \phi\bar\xi + Z$$

for $\phi$ a positive function on $L$ and $Z$ a tangent vector field, then compute, using a frame $X_i$ of the tangent space of $L$, that

$$\phi = \left(\frac{\det \bar h(X_i,X_j)}{\det(X_1,\dots,X_n,\bar\xi)^2}\right)^{\frac{1}{n+2}}, \tag{1}$$

$$Z^j = -\bar h^{ij}(X_i\phi + \phi\bar\tau_i), \tag{2}$$

for $\bar h^{ij}$ the inverse of $\bar h_{ij}$. □

*Remark.* Given a choice of orientation, the affine normal can be defined on any $C^3$ nonconvex hypersurface as long as the second fundamental form is nondegenerate.

Following Blaschke [7], we can use the affine normal to define other affine invariants on $L$. In particular, we have the Gauss and Weingarten formulas:

$$D_X Y = \nabla_X Y + h(X,Y)\xi, \tag{3}$$

$$D_X \xi = -S(X). \tag{4}$$

Here $\nabla$ is the induced *Blaschke connection*, $h$ is the *affine metric*, or *affine second fundamental form*, and $S$ is variously called the *affine shape operator*, the *affine third fundamental form*, or the *affine curvature*. The *affine mean curvature* is given by $H = \frac{1}{n}\mathrm{tr}S$.

The cubic tensor, given by $C = \nabla - \hat\nabla$ for $\hat\nabla$ the Levi-Civita connection of $h$, is another important invariant. Its main properties are the following:

**Proposition 2.** (1) *Apolarity: The trace* $\mathrm{tr}\,C = 0$ *(in index notation,* $C^i_{ij} = 0$*).*
(2) *Symmetry:* $C_{ijk} = h_{il}C^l_{jk}$ *is totally symmetric on all three indices.*
(3) *If the cubic form vanishes identically on* $L$*,* $L$ *is an open subset of a hyperquadric.*

The last item is due to Maschke (for analytic surfaces), Pick (for all surfaces), and Berwald (in general). See e.g. [59].

The symmetry of the cubic form is equivalent to $\nabla h$ being totally symmetric. The Ricci tensor of the affine metric on an affine sphere is of the form

$$R_{ij} = (n-1)Hg_{ij} + C_i^{k\ell}C_{jk\ell}. \tag{5}$$

Thus the Ricci tensor is always bounded from below by $(n-1)H$. This lower bound is essential in applying the maximum principle on complete affine manifolds.

As mentioned above, a proper affine sphere is a hypersurface whose affine normal lines all converge to a single point, the *center*. An improper affine sphere is a hypersurface whose affine normals are all parallel. There is an alternate definition in terms of the affine shape operator: An *affine sphere* is a hypersurface whose shape operator is a multiple of the identity $S = H\,I$, where $H$ is the affine mean curvature. Integrability conditions then force $H$ to be a constant. The sign of $H$

determines the type of affine sphere: If $H > 0$, $L$ is an elliptic affine sphere. If $H = 0$, $L$ is a parabolic affine sphere, and, if $H < 0$, $L$ is a hyperbolic affine sphere. It is often convenient to scale proper affine spheres in $\mathbb{R}^{n+1}$ to make the affine mean curvature $H = \pm 1$.

## 3. Examples

The prime examples of affine spheres are the quadric hypersurfaces. An ellipsoid has affine metric of constant positive curvature. This is to be expected, as the ellipsoid is affinely equivalent to a Euclidean sphere, and the isometries of the Euclidean sphere pull back to affine actions on the ellipsoid. Similarly, the affine metric on the hyperboloid has constant negative curvature, reflecting the Lorentz group action. An elliptic paraboloid in $\mathbb{R}^{n+1}$ admits a flat affine metric, and in fact the group of affine actions preserving the paraboloid is isomorphic to the group of isometries of $\mathbb{R}^n$.

As we will discuss below, quadrics are the only global examples of elliptic and parabolic affine spheres. There are many global hyperbolic affine spheres, asymptotic to each regular convex cone in $\mathbb{R}^{n+1}$ (a regular convex cone is an open convex cone which contains no lines) [10, 13, 15]. Ţiţeica already produced the example

$$\{(x^1, x^2, x^3) : x^1x^2x^3 = c > 0, x^i > 0\}$$

in $\mathbb{R}^3$, which is a hyperbolic affine sphere asymptotic to the boundary of the cone consisting of the first octant. Calabi [10] showed the corresponding example in $\mathbb{R}^{n+1}$ is a hyperbolic affine sphere.

$$\left\{ (x^1, \dots, x^{n+1}) : \prod_{i=1}^{n+1} x^i = c, \quad x^i > 0 \right\}. \tag{6}$$

The affine metric of Calabi's example is flat, and its cubic form never vanishes.

Calabi constructs his example via a product construction for hyperbolic affine spheres: If $L' \subset \mathbb{R}^{p+1}$ and $L'' \subset \mathbb{R}^{q+1}$ are hyperbolic affine spheres centered at the origin then the set

$$L = \left\{ \left( x' e^{\frac{-t}{p+1}}, x'' e^{\frac{t}{q+1}} \right) : x' \in L', x'' \in L'', t \in \mathbb{R} \right\}$$

is a hyperbolic affine sphere in $\mathbb{R}^{p+q+2}$ (though the affine mean curvature of $L$ is scaled by a complicated constant). Applying this product construction repeatedly to the zero-dimensional hyperbolic affine sphere $\{1\} \subset \mathbb{R}$ leads to Calabi's example in the first orthant of $\mathbb{R}^{n+1}$.

We also remark here that in many special cases, geometric conditions can be imposed on affine spheres (whether convex or not) to characterize specific equations. There is by now quite a large body of literature along these lines. For example, part III of Nomizu-Sasaki [59] details some examples, of which we mention a few. Magid-Ryan [57] classify all flat affine spheres (convex or not) in $\mathbb{R}^3$. Convex affine spheres with whose affine metrics have constant sectional curvature are shown to be quadrics or affine images of Calabi's example in Vrancken-Li-Simon [79]. An analogous question in the non-convex case was settled by Vrancken [78].

## 4. Two-dimensional affine spheres and Ţiţeica's equation

Ţiţeica first studied affine spheres in $\mathbb{R}^3$ (more properly, he studied a subset of affine spheres he called $S$-surfaces) [75, 76], and found conditions under which these surfaces can be integrated from initial data. More specifically, if $\alpha, \beta$ are two real parameters, and $v = v(\alpha, \beta)$ satisfies

$$\frac{\partial^2 v}{\partial\alpha\partial\beta} = e^v - e^{-2v}, \tag{7}$$

then the system of equations

$$\frac{\partial^2 f}{\partial\alpha^2} = \frac{\partial v}{\partial\alpha}\frac{\partial f}{\partial\alpha} + e^{-v}\frac{\partial f}{\partial\beta}, \tag{8}$$

$$\frac{\partial^2 f}{\partial\beta^2} = e^{-v}\frac{\partial f}{\partial\alpha} + \frac{\partial v}{\partial\beta}\frac{\partial f}{\partial\beta}, \tag{9}$$

$$\frac{\partial^2 f}{\partial\alpha\partial\beta} = e^v f \tag{10}$$

is integrable for $f = f(\alpha, \beta)$ a map into $\mathbb{R}^3$. This system can considered as a first-order system in the frame $\{f, \frac{\partial f}{\partial\alpha}, \frac{\partial f}{\partial\beta}\}$, and Ţiţeica's equation (7) is the integrability condition. More specifically, if $f\colon \mathcal{D} \to \mathbb{R}^3$ from a simply connected domain $\mathcal{D} \subset \mathbb{R}^2$, we may specify $f(x_0), f_\alpha(x_0), f_\beta(x_0)$ for any point $x_0 \in \mathcal{D}$. Equations (8-10) can be considered as a first-order system of PDEs in $\{f, f_\alpha, f_\beta\}$, and thus can be integrated along any path from $x_0$. Then (7) shows that the solution at any $x \in \mathcal{D}$ is independent of the path chosen from $x_0$ to $x$. The integrability condition is determined by checking e.g. $(f_{\alpha\beta})_\alpha = (f_{\alpha\alpha})_\beta$ and using the Frobenius Theorem.

In modern language, the surface parametrized by $f$ is then a proper, nonconvex affine sphere centered at the origin. This means the affine metric is indefinite. But as is often the case in the theory of two-dimensional integrable systems, the signature of the metric can be changed by considering complex parameters. So we can produce convex as well as nonconvex affine spheres by Ţiţeica's method. In fact, Ţiţeica produces the affine sphere

$$\{(x^1,x^2,x^3):x^1x^2x^3=1,x^i>0\}.$$

Wang [80] and Simon-Wang [67] have extended Ţiţeica's technique to all convex affine spheres in $\mathbb{R}^3$. The affine metric is positive definite on a convex surface in $\mathbb{R}^3$, and so there is an induced conformal structure. So in this case, we may assume that $f\colon\mathcal{D}\to\mathbb{R}^3$, where $\mathcal{D}\subset\mathbb{C}$ is simply connected and so that the map is a conformal map with respect to the affine metric on the image $f(\mathcal{D})$. Choose a local complex coordinate $z$ on $\mathcal{D}$, so that the affine metric is

$$e^{\psi}|dz|^2.$$

Then the apolarity condition on the cubic form shows that all but two components of the cubic form vanish. In terms of complex coordinates, if we set

$$U=C^{\bar 1}_{11}e^{\psi},$$

then the structure equations (3-4) for the (complexified) frame $\{f_z,f_{\bar z},\xi\}$ become

$$\begin{aligned}
f_{zz} &= \psi_z f_z+Ue^{-\psi}f_{\bar z},\\
f_{\bar z\bar z} &= \bar U e^{-\psi}f_z+\psi_{\bar z}f_{\bar z},\\
f_{z\bar z} &= \tfrac12 e^{\psi}\xi,\\
\xi_z &= -Hf_z,\\
\xi_{\bar z} &= -Hf_{\bar z}.
\end{aligned}$$

The integrability conditions for these equations are then

$$U_{\bar z} = 0, \tag{11}$$

$$\psi_{z\bar z}+|U|^2e^{-2\psi}+\frac{H}{2}e^{\psi} = 0. \tag{12}$$

If the holomorphic coordinate $z$ is changed, $U$ transforms as a cubic differential, which is holomorphic by (11). On a Riemann surface, (11) becomes for $e^{\psi}|dz|^2=e^{\phi}g$,

$$\Delta\phi+4\|U\|^2e^{-2\phi}+2He^{\phi}-2\kappa, \tag{13}$$

where $\Delta$ is the Laplace operator of $g$, $\|\cdot\|$ is the induced norm on cubic differential, and $\kappa$ is the Gauss curvature. Below we will discuss global solutions to (13) on Riemann surfaces due to [40, 41, 54, 55, 50, 51, 80], and their application to the geometry of projective, affine, and Calabi-Yau manifolds.

(We note different versions of Ţiţeica's equation also occur in other geometric contexts in which real forms of $\mathbf{SL}(3,\mathbb{C})$ act. For example, see McIntosh [58] for an application of solutions to

$$\psi_{z\bar z} - e^{-2\psi} + e^{\psi} = 0$$

to minimal Lagrangian immersions in $\mathbb{CP}^2$ and special Lagrangian cones in $\mathbb{C}^3$.)

The structure equations (3-4) are a first-order linear PDE system in the frame $\{\xi, f_z, f_{\bar z}\}$, and if (11 - 12) are satisfied, then (3-4) can be solved as an initial value problem on any simply connected domain $\mathcal{D}\subset\mathbb{C}$. In other words, for any $z_0\in\mathcal{D}$, if $\xi(z_0), f_z(z_0), f_{\bar z}(z_0)$ are specified, then (3-4) determine the frame at every point in $\mathcal{D}$.

For affine spheres, this determines an affine or projective holonomy action. Proper affine spheres with center at the origin naturally have holonomy in $\mathbf{SL}(3,\mathbb{R})$ (affine actions in $\mathbf{SA}(3,\mathbb{R})$ fixing the origin), while improper affine spheres with affine normal $(0,1)$ naturally have holonomy in $\mathbf{SA}(2,\mathbb{R})$ (actions in $\mathbf{SA}(3,\mathbb{R})$ fixing the affine normal). For the special case of a holomorphic equivalence relation $z\sim z+1$, the frame $\{\xi, f_z, f_{\bar z}\}$ is a well-defined frame on the Riemann surface $\mathcal{D}/\sim$. For example, for a hyperbolic affine sphere with center 0 and affine mean curvature $-1$, the affine normal $\xi = f$, and integrating the structure equations along a path from $z_0$ to $z_0+1$ calculates holonomy map in $\mathbf{SL}(3,\mathbb{R})$ for the frame $\{f, f_z, f_{\bar z}\}$. In particular, on a Riemann surface with a point singularity, it is often possible to prescribe the behavior of the affine metric and cubic differential near the singularity, and then to use the theory of ODEs to determine the conjugacy class of holonomy of a loop around the singularity. See [54, 55].

Parabolic affine spheres in $\mathbb{R}^3$ have much better integrability properties. Much like minimal surfaces in $\mathbb{R}^3$, there are Weierstrass formulas for reproducing parabolic affine spheres in terms of holomorphic functions.

The basic ideas leading to this Weierstrass formula have been available for quite a long time. Blaschke [7] recognizes that parabolic affine spheres with affine normal $(0,0,1)$ are locally given in terms of a graph of a convex function $u$ satisfying the Monge-Ampère equation

$$\det\frac{\partial^2 u}{\partial x^i\partial x^j} = 1, \tag{14}$$

and he shows that parabolic affine spheres in $\mathbb{R}^3$ are completely integrable, by using the observation in Darboux [22] that solutions to (14) in dimension two can locally be transformed into harmonic functions. Jörgens [37] uses the relation between solutions to (14) and complex analytic functions to show that any entire convex solution to (14) in $\mathbb{R}^2$ must be a quadratic polynomial. Moreover, Jörgens uses a natural transformation between parabolic affine two-spheres and minimal surfaces to reprove Bernstein's Theorem that any minimal surface in $\mathbb{R}^3$ which is an entire graph is a plane. See also Chapter 9 of Spivak [68].

The affine Weierstrass formula for affine maximal surfaces in $\mathbb{R}^3$ (a generalization of a parabolic affine sphere) is given in by Calabi [12], Terng [70], and Li [42], for affine maximal surfaces in $\mathbb{R}^3$. A similar description for parabolic affine spheres is given later by Ferrer-Martínez-Milán [27], although an equivalent formulation motivated by string theory is found earlier in Greene-Shapere-Vafa-Yau [34]. The affine Weierstrass formula of [27] is the following: Given two holomorphic functions $F$ and $G$ satisfying $|dF| < |dG|$ on a simply connected domain $\mathcal{D} \subset \mathbb{C}$, the parametrized surface

$$\left(\frac{1}{2}(G+\bar{F}), \frac{1}{8}(|G|^2 - |F|^2), \frac{1}{4}\mathrm{Re}\,(GF) - \frac{1}{2}\mathrm{Re}\int F\,dG\right)$$

is a parabolic affine sphere with affine normal $(0,0,1)$.

There is a higher-dimensional generalization of the Weierstrass representation formula to special Kähler manifolds. A special Kähler manifold is a Kähler manifold which admits a torsion-free, flat connection $\nabla$ with respect to which the Kähler form is parallel and so that $d_\nabla I = 0$ for $I$ the complex structure tensor [28]. Cortés [21] has shown that any special Kähler manifold may be constructed from the graph of a holomorphic function on a domain in $\mathbb{C}^n$. Z. Lu has shown that any complete special Kähler manifold is flat [56]. Lu's result also follows from Calabi's Theorem 2 below, since each special Kähler manifold carries the structure of a parabolic affine sphere [5].

It is also possible to relate solutions to other versions of Ţiţeica's equation to integrable systems, as Dunajski-Plansangkate have recently related radially symmetric solutions of (12) with $H = 1$ to solutions of Painlevé III.

## 5. Monge-Ampère Equations and Duality

In this section, we recount the real Monge-Ampère equations related to convex affine spheres, and use the conormal map and the Legendre transform to find dual affine spheres. In this context, the equation for a parabolic affine sphere, $\det u_{ij} = 1$, goes back to Blaschke. The equations for proper affine spheres are due to Calabi [10]; although we will primarily present them in the context of Gigena [30].

For simplicity, we state the following theorem only for affine spheres of affine mean curvature $H = -1, 0, 1$. More general proper affine spheres may be obtained by scaling.

**Proposition 3.**
- *A hyperbolic affine sphere with center 0 and affine mean curvature $-1$ is locally given by the radial graph of $-1/u$,*

$$\left\{ -\frac{1}{u(t)}(1, t^1, \dots, t^n) : t = (t^1, \dots, t^n) \in \Omega. \right\},$$

*where $\Omega$ is a domain in $\mathbb{R}^n$ (thought of as an inhomogeneous domain in $\mathbb{RP}^n$), and $u$ is a convex negative function satisfying*

$$\det \frac{\partial^2 u}{\partial t^i \partial t^j} = \left(-\frac{1}{u}\right)^{n+2}.$$

*The affine metric is given by*

$$-\frac{1}{u}\, \frac{\partial^2 u}{\partial t^i \partial t^j}\, dt^i dt^j.$$

- *A parabolic affine sphere with affine normal $(0, \dots, 0, 1)$ is given by the graph of a convex function $u$*

$$\{(x^1, \dots, x^n, u(x)) : x = (x^1, \dots, x^n) \in \mathcal{O} \subset \mathbb{R}^n$$

*satisfying the Monge-Ampère equation*

$$\det u_{ij} = 1.$$

*The affine metric is given by*

$$\frac{\partial^2 u}{\partial x^i \partial x^j}\, dx^i dx^j.$$

- *An elliptic affine sphere with center 0 and affine mean curvature 1 is given by the radial graph of $1/u$,*

$$\left\{ \frac{1}{u(t)}(1, t^1, \dots, t^n) : t = (t^1, \dots, t^n) \in \Omega. \right\},$$

*where $\Omega$ is a domain in $\mathbb{R}^n$ (thought of as an inhomogeneous domain in $\mathbb{RP}^n$), and $u$ is a convex positive function satisfying*

$$\det \frac{\partial^2 u}{\partial t^i \partial t^j} = \left(\frac{1}{u}\right)^{n+2}.$$

*The affine metric is given by*

$$\frac{1}{u}\, \frac{\partial^2 u}{\partial t^i \partial t^j}\, dt^i dt^j.$$

For each convex affine sphere as above in $\mathbb{R}^{n+1}$, there is a dual affine sphere in $\mathbb{R}_{n+1}$, the dual vector space to $\mathbb{R}^{n+1}$. The construction is slightly different in the case of proper and improper affine spheres, but both constructions are related to the Legendre transform. Given a smooth strictly convex function $v\colon \Omega \to \mathbb{R}$ on a convex domain $\Omega \subset \mathbb{R}^n$, the Legendre transform function $v^*$ is defined by

$$v^* + v = x^i \, \frac{\partial v}{\partial x^i}.$$

The function $v^*$ is considered primarily as a convex function $v^* : \Omega^{*,v} \to \mathbb{R}$ with

$$\Omega^{*,v} = \left\{ \left( \frac{\partial v}{\partial x^i}(x) \right) \in \mathbb{R}_n : x \in \Omega \right\}.$$

The duality of a parabolic affine sphere comes directly from the Legendre transform, while for proper affine spheres, the duality is provided by the conormal map. Given a hypersurface $L \subset \mathbb{R}^{n+1}$ which is transverse to the position vector, the conormal map $N\colon L \to \mathbb{R}_{n+1}$ is given for $x \in L$ by

$$N\colon x \mapsto \ell, \qquad \ell(x) = 1, \quad \ell(T_x L) = 0.$$

The conormal map is naturally related to the Legendre transform by the following formulas: If $u = u(t^1, \dots, t^n)$ is convex, and $L$ is the radial graph of $c/u$:

$$L = \{(c/u)(t^1, \dots, t^n, 1)\},$$

then the conormal map of $L$ is given by

$$\left\{ -\frac{1}{c} \left( -\frac{\partial u}{\partial t^1}, \dots, -\frac{\partial u}{\partial t^n}, u^* \right) \right\} \tag{15}$$

for $u^*$ the Legendre transform of $u$. The duality result for parabolic affine spheres follows from the well-known duality of the Monge-Ampère equation $\det u_{ij} = 1$ under the Legendre transform, while the duality result for proper affine spheres is originally due to Calabi, and first appears in the literature in Gigena [29].

**Proposition 4.** *To each affine sphere in $\mathbb{R}^{n+1}$, there is a dual affine sphere of the same type in the dual space $\mathbb{R}_{n+1}$. The dual of a proper affine sphere centered at the origin is given by the image of the conormal map.*

*For an improper affine sphere with affine normal $\xi = (0, \dots, 0, 1)$ given by the graph $(x, u(x))$, the dual is the graph of the Legendre transform of $u$.*

*Remark.* Proposition 4 and the relationship between the conormal map and the Legendre transform (15) lead to Calabi's original formulation of the relationship between proper affine spheres and solutions to the Monge-Ampère equation [10]: The graph of a convex function

$$\{(x, \psi(x)) : x \in \Omega \subset \mathbb{R}^n\}$$

is a proper affine sphere of affine mean curvature $H$ and center 0 if and only if the Legendre transform $\psi^*$ of $\psi$ satisfies

$$\det \psi^*_{ij} = (H\psi^*)^{-n-2}.$$

We need the notion of conjugate connections to explain the duality further. Given a smooth vector bundle $E \to M$, equipped with a nondegenerate metric $h$ and a connection $\nabla$, the *conjugate connection* $\nabla^*$ on $E$ is given by

$$d\, h(s,t) = h(\nabla s, t) + h(s, \nabla^* t)$$

for smooth sections $s, t$ of $E$. In the case $\nabla h$ is totally symmetric, which is always true for affine structures, the Levi-Civita connection $\hat{\nabla}$ of $h$ satisfies

$$\hat{\nabla} = \tfrac{1}{2}(\nabla + \nabla^*). \tag{16}$$

See e.g. [59].

**Proposition 5.** *The Blaschke connection on an improper affine sphere is flat. The Blaschke connection on a proper affine sphere is projectively flat.*

*In all cases, the dual map is an isometry with respect to the affine metrics. It maps the cubic form to its negative; in other words, the Blaschke connections of the dual hyperspheres are conjugate with respect to the affine metric.*

We sketch the proofs of Propositions 4 and 5 together.

*Sketch of proof.* For improper affine spheres, it is well known that the Legendre transform of a convex solution to the Monge-Ampère equation $\det u_{ij} = 1$ solves the same equation. Moreover, the affine metric for a parabolic affine sphere is of Hessian type $u_{ij}dx^i dx^j$, and the affine connection is flat. For such metrics, the conjugate connection is also flat [2], and corresponds to the dual affine structure.

For any convex affine hypersurface $L$ in $\mathbb{R}^{n+1}$, the image of the conormal map is naturally a centroaffine hypersurface $L^* \subset \mathbb{R}_{n+1}$ (in other words, $L^*$ is transverse to the position vector). The centroaffine connection of $L^*$ is conjugate to the affine (Blaschke) connection of $L$

[64, 59]. For $f$ the position vector of $L^*$, the centroaffine connection $\nabla^c$ is defined by the Gauss equation

$$D_X Y = \nabla^c_X Y + h^c(X,Y) f$$

defined by the splitting $T_f\mathbb{R}_{n+1} = T_f L^* + \langle f \rangle$. But since the affine normal for proper affine spheres with center 0 and affine mean curvature $\pm 1$ is equal to $\mp f$, $L^*$ is again a proper affine sphere with Blaschke connection $\nabla^c$.

Note that (16), by the definition of the cubic form, shows that passing from the Blaschke connection to its conjugate corresponds to mapping the cubic form to its negative. □

## 6. Global classification of affine spheres

The affine metric of a convex affine hypersurface in $\mathbb{R}^{n+1}$ is a Riemannian metric, and thus gives an affine-invariant notion of completeness on the hypersurface. We may also consider whether the convex hypersurface is properly embedded, which corresponds to an extrinsic notion of *Euclidean completeness* for hypersurfaces. In general, the two notions of completeness are different (see e.g. examples in [46]), but for affine spheres, the two notions are the same. This was conjectured by Calabi and proved, in the hyperbolic case, by Cheng-Yau.

**Theorem 1** (Jörgens, Calabi, Pogorelov, Cheng-Yau)**.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $u\colon \Omega \to \mathbb{R}$ be a smooth convex function satisfying the Monge-Ampère equation*

$$\det u_{ij} = 1$$

*and so that the graph*

$$\{(x, u(x)) : x \in \Omega\}$$

*is closed in $\mathbb{R}^{n+1}$. Then $u$ is a quadratic polynomial.*

This theorem is due to Jörgens originally for entire solutions when $n = 2$ [37], to Calabi for entire solutions for $n = 3, 4, 5$ [9], and Pogorelov for entire solutions for general $n$ [60]. Cheng-Yau proved the full theorem in [15]. In terms of affine geometry, this theorem is equivalent to the following:

**Corollary 6.** *Every convex properly embedded parabolic affine sphere in $\mathbb{R}^{n+1}$ is an elliptic paraboloid.*

The corresponding theorem for affine-complete parabolic affine spheres is due to Jörgens in dimension two [37] and Calabi in general [9]:

**Theorem 2.** *Any affine-complete parabolic affine sphere in* $\mathbb{R}^{n+1}$ *is an elliptic paraboloid.*

The proof is to use a maximum principle on noncompact manifolds to show that the norm of the cubic form must vanish. Then Berwald's theorem shows that the affine sphere must be a quadric hypersurface.

Similarly, all global examples of elliptic affine spheres must be ellipsoids. Blaschke first shows that any compact elliptic affine sphere in $\mathbb{R}^3$ is an ellipsoid [7], and Deicke later extends this theorem to any dimension [24].

Any affine-complete elliptic affine sphere must also be an ellipsoid. Calabi [10] shows that the Ricci curvature of an elliptic affine sphere is positive, and thus Myers's Theorem shows the affine sphere must be compact. Euclidean-complete elliptic affine spheres are affine complete by estimates of Cheng-Yau [15]. Thus any affine or Euclidean complete affine sphere in $\mathbb{R}^{n+1}$ must be compact, and thus is an ellipsoid.

For any strictly convex smooth affine hypersurface in $\mathbb{R}^{n+1}$, Trudinger-Wang prove that affine completeness implies Euclidean completeness if $n \geq 2$ [73].

## 7. Hyperbolic affine spheres and invariants of convex cones

Let $\mathbb{R}_{n+1}$ denote the dual vector space of $\mathbb{R}^{n+1}$. Then given an open convex cone $\mathcal{C} \subset \mathbb{R}^{n+1}$, the dual cone $\mathcal{C}^* \subset \mathbb{R}_{n+1}$ can be defined as

$$\mathcal{C}^* = \{\ell \in \mathbb{R}_{n+1} : \ell(x) > 0 \text{ for all } x \in \mathcal{C}\}.$$

The existence of hyperbolic affine spheres is due to Cheng-Yau. In [13], they show that for any convex bounded domain $\Omega \subset \mathbb{R}^n$, there is a unique convex solution to the Dirichlet problem

$$\det u_{ij} = \left(-\frac{1}{u}\right)^{n+2}, \qquad u = 0 \text{ on } \partial\Omega. \tag{17}$$

Let

$$\mathcal{C} = \{s(1,t) : t \in \Omega, s > 0\}$$

be the cone over $\Omega$. Then they show in [15] that $u$ induces the hyperbolic affine sphere asymptotic to the boundary of the dual cone $\mathcal{C}^*$ by taking the Euclidean graph of the Legendre transform of $u$. The description of the hyperbolic affine sphere asymptotic to $\mathcal{C}$ is given in Gigena [30] as the radial graph of $-1/u$.

We outline a proof of Cheng-Yau's solution to the Monge-Ampère equation (17). By now, producing convex solutions to Dirichlet boundary problems for real Monge-Ampère equations $\det v_{ij} = F(x, v)$ on strictly convex bounded domains for smooth positive functions $F$ is fairly standard (see e.g. [31, Theorem 17.22]). Equation (17) is singular in two ways, however. First of all, since $u=0$ on $\partial\Omega$, the right-hand side $(-\frac{1}{u})^{n+2}$ blows up on $\partial\Omega$. Second, $\Omega$ is allowed to be any convex domain with potentially no more than Lipschitz boundary regularity. Below we give limiting arguments, essentially due to Cheng-Yau [13], to produce solutions to (17) in the general case. Calabi's example (6) above provides an explicit solution to (17) on a simplex, which is used as a barrier. In dimension two, Loewner-Nirenberg [47] solved (17) on domains with strictly convex smooth boundary, and noted the projective invariance of solutions to (17) from a point of view independent of affine geometry.

*Sketch of proof.* Let $\Omega\subset\mathbb{R}^n$ be a bounded convex domain, and consider an exhaustion

$$\Omega=\bigcup_k \Omega_k, \qquad \Omega_k \Subset \Omega_{k+1},$$

for $\Omega_k$ strictly convex bounded domains with smooth boundary. Standard theory for the real Monge-Ampère equation shows that there is a unique convex solution $u_{k,\epsilon}$ to

$$\det u_{ij} = \left(-\frac{1}{u-\epsilon}\right)^{n+2} \text{ on } \Omega_k, \qquad u|_{\partial\Omega_k}=0 \tag{18}$$

for each $\epsilon>0$. Each $u_{k,\epsilon}$ is smooth on $\Omega_k$ and is continuous on $\bar\Omega_k$. Our solution $u$ will be the limit of $u_{k,\epsilon}$ as $k\to\infty$ and $\epsilon\to 0$.

First, we let $\epsilon\to 0$. The maximum principle shows that $u_{k,\epsilon}\le u_{k,\epsilon'}$ if $\epsilon<\epsilon'$: We will find a contradiction if the difference $w=u_{k,\epsilon}-u_{k,\epsilon'}$ is positive anywhere on $\bar\Omega_k$. Since $w$ vanishes on $\partial\Omega_k$, if $w$ is positive anywhere, it has a positive maximum point $p\in\Omega_k$. Then at $p$, the Hessian

$$\frac{\partial^2 w}{\partial x^i\partial x^j}(p)=\frac{\partial^2 u_{k,\epsilon}}{\partial x^i\partial x^j}(p)-\frac{\partial^2 u_{k,\epsilon'}}{\partial x^i\partial x^j}(p)$$

is negative definite. Since the Hessians of both $u_{k,\epsilon}$ and $u_{k,\epsilon'}$ are positive-definite, a simple lemma shows that

$$\det\left(\frac{\partial^2 u_{k,\epsilon}}{\partial x^i\partial x^j}(p)\right)\le\det\left(\frac{\partial^2 u_{k,\epsilon'}}{\partial x^i\partial x^j}(p)\right).$$

But then, the equation (18) shows

$$\left(-\frac{1}{u_{k,\epsilon}-\epsilon}(p)\right)^{n+2}\le\left(-\frac{1}{u_{k,\epsilon'}(p)-\epsilon'}\right)^{n+2}.$$

This contradicts the assumptions that $\epsilon < \epsilon'$ and $w(p) = u_{k,\epsilon}(p) - u_{k,\epsilon'}(p) > 0$.

Therefore, as $\epsilon \to 0$, $u_{k,\epsilon}$ is decreasing pointwise, and thus there is a pointwise limit function $u_k = u_{k,0}$ unless the sequence decreases to $-\infty$. By using affine transformations, Calabi's example provides an explicit solution $v$ to (17) on any simplex $\Delta \subset \mathbb{R}^n$. For a given boundary point $q \in \partial\Omega_k$, choose $\Delta$ so that $\Delta \supset \Omega_k$ and $q \in \partial\Delta \cap \partial\Omega_k$. The maximum principle then shows that $u_{k,\epsilon} \geq v$ on all of $\bar{\Omega}_k$, and so the limit function does not go to $-\infty$ and is continuous at the boundary point $q$. Estimates for the Monge-Ampère equation show the convergence $u_{k,\epsilon} \to u_k$ is in $C^\infty_{\mathrm{loc}}(\Omega_k)$ and thus $u_k$ solves (17): $C^1$ estimates are standard for convex functions, while Pogorelov provides interior $C^2$ estimates [60], and then one can use either the interior $C^3$ estimates of Calabi [9] or interior $C^{2,\alpha}$ estimates of Evans to achieve the desired regularity.

Now $u_k$ solves (17) on $\Omega_k$. Now the same ideas allow us to take $k \to \infty$: The maximum principle implies $u_{k+1} \leq u_k$ on $\Omega_k$ and Calabi's barriers ensures the limit $u_k \to u$ is finite and continuous to the boundary of $\Omega$. Interior estimates as mentioned above implies the convergence $u_k \to u$ is in $C^\infty_{\mathrm{loc}}(\Omega)$, and so $u$ solves (17) on $\Omega$. □

The following theorem is due to Cheng-Yau [13, 15] and Calabi-Nirenberg (unpublished), with clarifications due to Gigena [30], Sasaki [61] and A.-M. Li [43, 44].

**Theorem 3.** *For any open convex cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ which contains no lines, there is a unique convex properly embedded hyperbolic affine sphere $L \subset \mathbb{R}^{n+1}$ which has affine mean curvature $-1$, has the vertex of $\mathcal{C}$ as its center, and is asymptotic to the boundary $\partial C$. For any immersed hyperbolic affine sphere $L \to \mathbb{R}^{n+1}$, the properness of the immersion is equivalent to the completeness of the affine metric, and any such $L$ is a properly embedded hypersurface asymptotic to the boundary of the cone $\mathcal{C}$ given by the convex hull of $H$ and its center.*

*The image of the conormal map of $L$ is the unique hyperbolic affine sphere of affine mean curvature $-1$ vertex 0 asymptotic in the dual space $\mathbb{R}_{n+1}$ to the boundary of the dual cone $\mathcal{C}^*$.*

*Sketch of proof.* Cheng and Yau [15] prove (with a small gap) that an affine sphere is properly embedded if and only if its affine metric is complete. Moreover, any such hyperbolic affine sphere $L$ is asymptotic to the cone given the convex hull of $L$ and its center. (Calabi-Nirenberg, in unpublished work, establish the same result.) In [43, 44], A.-M. Li clarified the proof of Cheng-Yau by using essentially the same estimates developed in [15] to show that affine completeness implies Euclidean completeness for hyperbolic affine spheres. Trudinger-Wang have recently proved that any convex affine-complete hypersurface in $\mathbb{R}^{n+1}$ is Euclidean complete if $n \geq 2$ [73].

Given a bounded convex domain $\Omega \subset \mathbb{R}^n$, considered as lying in an affine subspace $\{x^0 = 1\} \subset \mathbb{R}^{n+1}$, denote the cone over $\Omega$ as

$$\mathcal{C} = \bigcup_{s>0} \{s(1,t) : t \in \Omega\}.$$

Consider Cheng-Yau's solution to the Monge-Ampère equation $\det u_{ij} = (-\frac{1}{u})^{n+2}$ with zero Dirichlet boundary values [13]. The hyperbolic affine sphere $L$ in the cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ given by the radial graph of $-1/u$ is then obviously asymptotic to the boundary of $\mathcal{C}$ and is Euclidean complete, by work of Gigena [30]. (This formulation was known experts in the 1970s, when Cheng-Yau completed their work.) An alternate approach is taken by Sasaki [61], who shows that the dual affine sphere to $L$, given by the Legendre transform of $u$, is a Euclidean complete hyperbolic affine sphere $L^*$ asymptotic to the dual cone $\mathcal{C}^* \subset \mathbb{R}_{n+1}$.

These two affine spheres $L$ and $L^*$ are dual under the conormal map. The local statement of duality follows from the identification of the centroaffine connection of the image of the conormal map with the conjugate to the Blaschke connection, already found in the Schirokovs' book [64]. In other words, the dual of $L$ is a hyperbolic affine sphere. To show that the image of the conormal map of $L$ is $L^*$ follows from Gigena [30]. Alternately, once we produce either $L$ or $L^*$ as a Euclidean-complete affine sphere by [30] or [61], it must be affine complete. Since the conormal map is an isometry of the affine metric, its dual $L^*$ or $L$ respectively is also affine complete, and so it must be Euclidean complete. We can identify this dual with the correct target by the uniqueness of solutions to (17), which follows from the maximum principle.

We also indicate very briefly some of Cheng-Yau's estimates [15]: To show a Euclidean-complete affine sphere $L$ is affine-complete, first of all note that we can choose affine coordinates on $\mathbb{R}^{n+1}$ so that one coordinate function (a height function $\mathcal{H}$) is proper on $L$. Then on the compact domain $L_c = \{\mathcal{H} \leq c\}$, use the maximum principle to estimate functions of the form

$$\exp\left(\frac{-m}{c-\mathcal{H}}\right)\frac{|\hat{\nabla}\mathcal{H}|^2}{(\mathcal{H}+\epsilon)^p}$$

for appropriate constants $m, \epsilon, p$ and $\hat{\nabla}$ the Levi-Civita connection of the affine metric. Then, taking $c \to \infty$ produces an estimate of the form

$$\frac{|\hat{\nabla}\mathcal{H}|}{\mathcal{H}+1} \leq C$$

for a constant $C$. Since $\log(\mathcal{H}+1)$ is proper on $L$, this gradient estimate shows the affine metric is complete.

To prove that affine completeness implies Euclidean completeness for hyperbolic affine spheres, a gradient estimate for the Legendre transform of $\mathcal{H}$ [15, 43, 44] and an estimate of the norm of the cubic form [10] are used. □

Calabi shows that the affine metric on complete hyperbolic affine spheres has Ricci curvature bounded between 0 and a negative constant [10]. The lower bound is a pointwise formula for the Ricci tensor (5), while the fact that the Ricci is nonpositive requires global techniques from Riemannian geometry, and a bound on the norm of the cubic form. We note that the extreme cases are both satisfied by homogeneous affine spheres: The affine metric on a hyperboloid has constant negative sectional curvature, while the affine metric on Calabi's example (6) is flat. It is instructive to think of these examples in terms of extrema of convex projective domains. The hyperboloid is asymptotic to a cone over a round ball, while Calabi's example is asymptotic to a cone over a simplex.

The Monge-Ampère equation (17) is very similar to Fefferman's equation for complete Kähler-Einstein metrics on bounded smooth strictly pseudoconvex domains in $\mathbb{C}^n$ [62]. Sasaki has used a similar asymptotic calculations to Fefferman's to compute projective invariants of convex domains [63].

## 8. Projective Manifolds

An $\mathbb{RP}^n$ structure (real projective structure) on a smooth manifold $M$ is given by an atlas of coordinate charts in $\mathbb{RP}^n$ with gluing maps locally constant projective maps in $\mathbf{PGL}(n+1,\mathbb{R})$. (So in Thurston's language, an $\mathbb{RP}^n$ manifold is an $(X,G)$ manifold for the homogeneous space $X=\mathbb{RP}^n$ with group $G=\mathbf{PGL}(n+1,\mathbb{R})$.) An $\mathbb{RP}^n$ manifold $M$ is *properly convex* if it is given by a quotient

$$M = \Omega/\Gamma,$$

where $\Omega$ is a bounded convex domain in $\mathbb{R}^n \subset \mathbb{RP}^n$ and $\Gamma \subset \mathbf{PGL}(n+1,\mathbb{R})$ acts discretely and properly discontinuously on $\Omega$. Hyperbolic manifolds all admit $\mathbb{RP}^n$ structures by the Klein model of hyperbolic space: In this model hyperbolic space $\mathbb{H}^n$ is represented by an open ball $B \subset \mathbb{R}^n$, and the hyperbolic isometries are exactly the projective actions on $\mathbb{RP}^n \supset \mathbb{R}^n$ which act on $B$.

A *geodesic* in an $\mathbb{RP}^n$ manifold is a path which is a straight line segment in each $\mathbb{RP}^n$ coordinate chart. This leads to an alternate definition of an $\mathbb{RP}^n$ structure: An $\mathbb{RP}^n$ structure on $M$ is given by a projective equivalence class of projectively flat connections on the tangent bundle $TM$. Two connections on $TM$ are said to be *projectively equivalent* if they have the same geodesics, when considered as unparametrized sets. A connection is *projectively flat* if it is locally projectively equivalent to a flat, torsion-free connection. The relation between the two definitions of $\mathbb{RP}^n$ manifold is the following: The geodesics of the $\mathbb{RP}^n$ structure are then exactly the geodesics of each projectively flat connection in the equivalence class. (Yet another equivalent definition of an $\mathbb{RP}^n$ structure on a manifold $M$ is the existence of a flat projective *Cartan connection* on $M$. See e.g. Kobayashi [38].)

For a hyperbolic affine sphere $L$ with center 0 and affine mean curvature $-1$, Gauss's equation reads

$$D_X Y = \nabla_X Y + h(X,Y)f,$$

where $f$ is the position vector. In particular, in this case, the geodesics of $\nabla$ project to straight lines under the projection $L \to \Omega$ (see e.g. [59]). This shows that $\nabla$ is projectively flat. Note this argument shows that any centroaffine connection $\tilde\nabla$ is projectively flat, where $\tilde\nabla$ is defined by $D_X Y = \tilde\nabla_X Y + \tilde h(X,Y)f$ for any hypersurface transverse to the position vector $f$. A partial converse is also true ([59] or [26]):

**Proposition 7.** *Any manifold equipped with a projectively flat, torsion-free connection with symmetric Ricci tensor is local the pull-back of the centroaffine connection of a hypersurface under a diffeomorphism.*

An $\mathbb{RP}^n$ manifold $M$ admits a *development pair* of a developing map and holonomy. Given a universal cover $\tilde M$ of $M$ and a fundamental group $\pi_1 M$ corresponding to a base point $x_0$, there is a pair (dev, hol) of $\mathrm{dev} : \tilde M \to \mathbb{RP}^n$ and $\mathrm{hol} :\to \mathbf{PGL}(n+1,\mathbb{R})$ satisfying $\mathrm{dev}\circ\mathrm{hol}(\gamma) = \gamma\circ\mathrm{dev}$ for all $\gamma \in \pi_1 M$. The pair dev is unique up to composition in dev and conjugation in hol. An analog of this theorem is valid for all $(X,G)$ manifolds, and is due to Ehresmann. See e.g. Goldman [32]. The developing map is constructed by starting at the base point $x_0 \in M$ with a projective coordinate chart around $x_0$. Then along any path in $\tilde M$ from a lift $\tilde x_0$ of $x_0$, there is a unique choice of coordinate charts. This defines the developing map $\mathrm{dev} : \tilde M \to \mathbb{RP}^n$, and the holonomy $\mathrm{hol}(\gamma)$ is given by the coordinate transformation in $\mathbf{PGL}(n+1,\mathbb{R})$ given by developing the path from $x_0$ to $\gamma(x_0)$.

The developing map of an $\mathbb{RP}^n$ manifold closely corresponds to the centroaffine hypersurface picture above. If $\nabla$ is a projectively flat connection satisfying the conditions of Proposition 7, then the projection of the centroaffine hypersurface above from $\mathbb{R}^{n+1}\setminus\{0\}$ to $\mathbb{RP}^n$ coincides with the developing map.

To each properly convex $\mathbb{RP}^n$ manifold, there is a dual manifold modeled on the dual projective space $\mathbb{RP}_n$. In particular, if $M=\Omega/\Gamma$, where $\Omega$ is a bounded domain in $\mathbb{R}^n\subset\mathbb{RP}^n$ corresponding to the regular convex cone $\mathcal{C}\subset\mathbb{R}^{n+1}$. Then $\Omega^*$ can be taken as the projection to $\mathbb{RP}_n$ of the dual cone $\mathcal{C}^*\subset\mathbb{R}_n$.

By the uniqueness and invariance of hyperbolic affine spheres, together with the duality result, we have

**Proposition 8.** *The conormal map between dual hyperbolic affine spheres provides a natural map from any properly convex $\mathbb{RP}^n$ manifold $M$ to its projective dual manifold $M^*$. This map is an isometry with respect to the affine metrics and it takes the Blaschke connection on $M$ to the conjugate of the Blaschke connection on $M^*$.*

If a properly convex $\mathbb{RP}^n$ manifold $M$ is written as $M=\Omega/\rho(\pi_1M)$ for $\Omega\subset\mathbb{R}^n$ a bounded convex domain and $\rho:\pi_1M\to\mathbf{PGL}(n+1,\mathbb{R})$ a holonomy representation of the fundamental group, then the *dual* $\mathbb{RP}^n$ *structure* on $M$ is given by $\Omega^*/\chi(\rho(\pi_1M))$ for $\chi$ the map from the projective linear group on $\mathbb{R}^{n+1}$ to the projective group on its dual $\mathbb{R}_{n+1}$ given $\chi(\gamma)=(\gamma^\top)^{-1}$. The previous proposition then shows that the conormal map of hyperbolic affine spheres provides a natural identification of $M$ with its dual $\mathbb{RP}^n$ manifold.

Given a convex bounded domain $\Omega\subset\mathbb{R}^n\subset\mathbb{RP}^n$ corresponds to a convex cone $\mathcal{C}\subset\mathbb{R}^{n+1}$. For the unique hyperbolic affine sphere $H$ asymptotic to the boundary of $\mathcal{C}$ then the projective quotient map $\mathbb{R}^{n+1}\setminus\{0\}\to\mathbb{RP}^n$ induces a diffeomorphism $H\to\Omega$. The affine invariants of $H$ then descend to any projective quotient of $\Omega$, and so provide invariants on any properly convex $\mathbb{RP}^n$ manifold. See e.g. [52]

**Proposition 9.** *On any properly convex $\mathbb{RP}^n$ manifold $M$, the unique hyperbolic affine sphere asymptotic to the cone over the universal cover of $M$ determines the following data: a complete Riemannian metric (the affine metric), and two canonical projectively flat connections (the affine connection $\nabla$ and the conjugate connection $\nabla^*=\nabla-2C$), representing the $\mathbb{RP}^n$ structure and the dual $\mathbb{RP}^n$ structure.*

*Proof.* The duality theorem above 5 shows that dual hyperbolic affine sphere has affine connection equal to the conjugate connection $\nabla^*=$

$\nabla - 2C$ for $C$ the cubic form. Both $\nabla$ and $\nabla^*$ are projectively flat, since they are centroaffine connections on hypersurfaces in $\mathbb{R}^{n+1}$. □

Jaejeong Lee has recently proved the following theorem by using hyperbolic affine spheres and their duality structure under the conormal map.

**Theorem 4.** *Every compact properly convex $\mathbb{RP}^n$ manifold $M$ has a fundamental domain given by a bounded polytope in $\mathbb{R}^n \subset \mathbb{RP}^n$.*

The case of convex $\mathbb{RP}^2$ structures is particularly rich, since Ţiţeica's equation provides a link between hyperbolic affine spheres and holomorphic cubic differentials on surfaces.

As in Section 4 above, Wang [80] provides a version of Ţiţeica's developing map for two-dimensional hyperbolic affine spheres. Given a Riemann surface $\Sigma$ with a holomorphic cubic differential $U$ and a complete conformal metric $e^u h$ satisfying

$$\Delta u + 4\|U\|^2 e^{-2u} - 2e^u - 2\kappa$$

as in Section 4 above, the affine structure equations produce a map from the universal cover $\tilde{\Sigma} \to \mathbb{R}^3$. On a compact Riemann surface equipped with a hyperbolic background metric, Wang's equation has a unique solution for each pair $(\Sigma, U)$ [80, 40, 52].

Thus we have the following theorem, due independently to Labourie [40, 41] and the author [52]:

**Theorem 5.** *On a closed oriented surface of genus $g \geq 2$, a convex $\mathbb{RP}^2$ structure is equivalent to a pair $(\Sigma, U)$ consisting of a conformal structure $\Sigma$ on the surface and a holomorphic cubic differential on $\Sigma$.*

One can define the deformation space of convex $\mathbb{RP}^2$ structures by analogously to Teichmüller space: On a closed oriented surface $R$ of genus $g \geq 2$, the deformation space $\mathcal{G}_g$ of convex $\mathbb{RP}^2$ structures can be defined as the set of equivalence classes of pairs $[S, f]$, where $S$ is a convex $\mathbb{RP}^2$ surface, $f \colon R \to S$ is an orientation-preserving diffeomorphism, and $(S, f) \sim (S', f')$ if and only if $f' \circ f^{-1}$ is homotopic to a diffeomorphism from $S \to S'$ which preserves the projective structure. See e.g. Goldman [33].

The Riemann-Roch theorem then shows that the deformation space of convex $\mathbb{RP}^2$ structures has the structure of the total space of the holomorphic vector bundle over the Teichmüller space of conformal structures on the surface whose fibers are the vector space of holomorphic cubic differentials.

**Corollary 10** (Goldman [33]). *The deformation space of convex* $\mathbb{RP}^2$ *structures on a closed oriented surface of genus* $g \geq 2$ *is a cell of real dimension* $16g - 16$.

We may define the moduli space of convex $\mathbb{RP}^2$ structures as the space of all projective diffeomorphism classes of convex $\mathbb{RP}^2$ structures on a closed oriented surface $R$ of genus $g \geq 2$. As in the case with Teichmüller space, this moduli space of convex $\mathbb{RP}^2$ structures is the quotient of the deformation space $\mathcal{G}_g$ by the mapping class group. Theorem 5 allows us to describe this space as an (orbifold) vector bundle over the moduli space of Riemann surfaces of genus $g$.

In general, it is not easy to determine explicitly the $\mathbb{RP}^2$ holonomy determined by a pair $(\Sigma, U)$ for $\Sigma$ a complex structure and $U$ a cubic differential. But in some limiting cases, we can determine some information.

We may determine the $\mathbb{RP}^2$ holonomy in limits of pairs $(\Sigma_t, U_t)$ for $\Sigma_t \to \Sigma_\infty$ a point in the Deligne-Mumford compactification of the moduli space of Riemann surfaces. At such a limit $\Sigma_\infty$, one or more necks of the Riemann surface are pinched to nodes. Holomorphic cubic differentials on $\Sigma_t$ then naturally have limits as regular cubic differentials on $\Sigma_\infty$, which are allowed to have poles of order 3 at each puncture. We may define the residue of the cubic differential to be the $dz^3/z^3$ coefficient of a regular cubic differential with pole at $z = 0$. This residue is invariant under the choice of local holomorphic coordinate $z$. For regular cubic differentials, the residues across each puncture must sum to zero. In [54], around sufficiently small loops around each puncture, the holonomy determined by the structure equations (3-4) is an ODE which approaches

$$\partial_x X = AX,$$

for a frame $X$ and a constant matrix $A$. This determines the conjugacy class as that of $\exp(A)$, at least when the holonomy has distinct eigenvalues. This gives an explicit relationship between the cubic differential and Goldman's analog of Fenchel-Nielsen's length coordinates [33].

An $\mathbb{RP}^2$ structure on a surface determines (up to conjugacy) a holonomy representation from $\pi_1 R \to \mathbf{SL}(3, \mathbb{R})$. Using the theory of Higgs bundles, Hitchin has identified a component of the representation space of the fundamental group of a closed surface $R$ of genus $g \geq 2$ into $\mathbf{SL}(3, \mathbb{R})$ (and into any split real form of a semisimple Lie group) [36]. Fixing a Riemann surface structure $\Sigma$ on $R$, Hitchin identifies the component of the representation space with $H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^3)$

the space of pairs of quadratic and cubic differentials over $\Sigma$. Choi-Goldman [18] show that the holonomy map identifies the space of convex $\mathbb{RP}^2$ structures on $R$ with Hitchin's component of the representation space. Under this map Labourie [41] has identified Hitchin's cubic differential with (a constant multiple of) C.P. Wang's cubic differential $U$ coming from the hyperbolic affine sphere: The $\mathbb{RP}^2$ structure coming from the affine sphere determined by $(\Sigma, -\frac{1}{12}U)$ has holonomy determined by Hitchin's Higgs bundle representation from the Riemann surface $\Sigma$ with quadratic differential 0 and cubic differential $U$.

Labourie [41] also notes that a hyperbolic affine sphere $L \subset \mathbb{R}^3$ naturally gives rise to a harmonic map into the symmetric space of metrics on $\mathbb{R}^3$, which may be identified with $\mathbf{SL}(3,\mathbb{R})/\mathbf{SO}(3,\mathbb{R})$. If $L \subset \mathbb{R}^3$ is a hyperbolic affine sphere with center 0 and affine mean curvature $-1$, and $p \in L$, then we may define the *Blaschke lift*

$$\mathcal{B}\colon L \to \mathbf{SL}(3,\mathbb{R})/\mathbf{SO}(3,\mathbb{R})$$

as the metric on $\mathbb{R}^3$ given by the orthogonal direct sum of the affine metric on the tangent space $T_pL$ and the metric on the line $\langle p \rangle$ for which the position vector $p$ has norm 1. Then with respect to the affine metric on $L$, $\mathcal{B}$ is a harmonic map. Under a projective quotient of $L$, $\mathcal{B}$ is a twisted harmonic map (a section of a bundle). Such a harmonic metric is a natural foundation of the theory of Higgs bundles (see Corlette [20]), and provides a direct link between affine spheres and Hitchin's Higgs bundle theory.

(We also note that if the affine sphere $L$ is globally asymptotic to the boundary of a convex cone $\mathcal{C} \subset \mathbb{R}^3$, then the Blaschke lift on $L$ is the restriction of Cheng-Yau's complete affine Kähler-Einstein metric on $\mathcal{C}$—see Sasaki [62] and the discussion in the next section.)

## 9. Affine Manifolds

An affine structure on a smooth manifold $M$ is provided by a maximal atlas of coordinate charts in $\mathbb{R}^n$ with locally constant affine transition maps. Thus, in terms of Thurston's notation, an affine structure is the structure of an $(X,G)$ manifold with $X = \mathbb{R}^n$ and

$$G = \mathbf{Aff}(n,\mathbb{R}) = \{\Phi\colon \mathbb{R}^n \to \mathbb{R}^n, \Phi(x) = Ax + b\}.$$

Equivalently, an affine structure is provided by a flat, torsion-free connection $\nabla$ on the tangent bundle: In this case, the geodesics of $\nabla$ are straight line segments in the affine coordinate charts on $M$.

Affine manifolds are related to parabolic affine spheres through the Monge-Ampère equation $\det \phi_{ij} = 1$ (recall the graph of such a $\phi$ is a parabolic affine sphere). The Blaschke connection of a parabolic affine sphere is torsion-free and flat. Moreover, under an affine change of coordinates, the Hessian of a function $\phi$ transforms as a tensor. In particular, a convex function $\phi$ defines a Riemannian metric $\frac{\partial^2\phi}{\partial x^i \partial x^j} dx^i dx^j$ on an affine manifold. A Riemannian metric on an affine manifold which is locally of this form is called an affine Kähler, or Hessian, metric. If $M$ admits a $\nabla$-parallel volume form $\nu$ (in other words, the affine holonomy is the special affine group $\mathbf{SA}(n,\mathbb{R})$), then the Monge-Ampère equation $\det \phi_{ij} = \nu^2$ can be written as $\det \phi_{ij} = 1$ in special affine coordinates.

The tangent bundle of an affine manifold naturally carries a complex structure. Let $x \mapsto Ax + b$ be an affine change of coordinates for affine coordinates $x = (x^1, \dots, x^n)$, and let $y = (y^1, \dots, y^n)$ denote frame coordinates on the tangent space by representing tangent vectors as $y^i \frac{\partial}{\partial x^i}$. Then $z^i = x^i + \sqrt{-1} y^i$ form complex coordinates on the tangent bundle via the gluing map $z \mapsto Az + b$. We may denote the tangent bundle $TM$ with this complex structure as $M^{\mathbb{C}}$. (Note this construction can also be seen as gluing together tube domains of the form $\Omega + \sqrt{-1}\mathbb{R}^n \subset \mathbb{C}^n$ where $\Omega$ is an affine coordinate chart.) An affine Kähler metric on $M$ induces a Kähler metric on $M^{\mathbb{C}}$, and if the metric satisfies the Monge-Ampère equation $\det \phi_{ij} = 1$, induces a Ricci-flat Kähler metric on $M^{\mathbb{C}}$.

This picture is important to the conjecture of Strominger-Yau-Zaslow [69] about mirror symmetry in string theory. At least for Calabi-Yau manifolds at or near certain degenerate limits in their moduli space (called large-complex-structure limits), a Calabi-Yau manifold $N$ of dimension $n$ is conjectured to be the total space of a singular fibration $\pi\colon N^\circ \to M$, where $M$ is conjectured to consist an affine manifold $M^{\rm reg}$ of dimension $n$, together with a singular locus of real codimension two. Over the regular points $M^{\rm reg}$, the fibers of $\pi$ are conjectured to be special Lagrangian tori in $N$. $N^{\rm reg} = \pi^{-1} M^{\rm reg}$ admits a Calabi-Yau structure as a quotient of the tangent bundle structure above: $M^{\rm reg}$ is an affine manifold equipped with an affine flat connection $\nabla$ and an affine Kähler metric satisfying the Monge-Ampère equation $\det u_{ij} = 1$. Therefore, the tangent bundle $TM^{\rm reg}$ admits a Calabi-Yau metric. If the linear part of the affine holonomy of $M^{\rm reg}$ is integral—conjugate to a map in $\mathbf{SL}(n,\mathbb{Z})$—then we may take the quotient $TM^{\rm reg}/\Lambda = N^{\rm reg}$ for $\Lambda$ a $\nabla$-invariant lattice in $TM^{\rm reg}$. This provides the Calabi-Yau structure on $N^{\rm reg}$.

The mirror conjecture of Strominger-Yau-Zaslow [69], at least in a simplified form, is that the mirror Calabi-Yau can be determined by the Legendre transform of the local semi-flat Calabi Yau potential $u$ on the affine manifold $M^{\rm reg}$. This is the same as the duality between parabolic affine spheres we outline in Propositions 4 and 5 above: the semi-flat Calabi-Yau metrics on the $M^{\rm reg}$ and its dual $M^{{\rm reg},*}$ are isometric, while the affine flat connections are conjugate to each other with respect to the affine metric. The conjecture of [69] also requires the torus quotients to be replaced by their dual tori, and there should also be instanton correction terms, which we will not discuss in this paper.

The structure of the singular locus $S = M \setminus M^{\rm reg}$ is more complicated, but the picture is largely understood in dimension two due to the work of Gross-Wilson [35]. In this case, K3 surfaces, when properly rescaled, degenerate to a semi-flat Calabi-Yau structure on $S^2$ minus 24 points. The author has produced many semi-flat Calabi-Yau structures on $\mathbb{CP}^1$ minus a finite number of points by using solving the PDE of Ţiţeica and Simon-Wang

$$\Delta u + 4e^{-2u}\|U\|^2 - 2\kappa = 0$$

for $U$ a holomorphic cubic differential on $\mathbb{CP}^1$ with poles of order one [55]. The affine metric and holonomy are asymptotically the same as those studied in [35]. Again, it is more difficult from this point of view to determine from the cubic differential the full affine holonomy from $\pi_1 M^{\rm reg} \to \mathbf{SA}(2,\mathbb{R})$.

In dimension three, the base manifold $M^{\rm reg}$ is conjectured to be a three-manifold minus a graph. Generically, one can assume the graph to have trivalent vertices, and a fundamental problem is to produce nontrivial semi-flat Calabi-Yau structures locally near a trivalent vertex (on a ball in $\mathbb{R}^3$ minus the "Y" vertex of a graph). In [50, 51], we construct such metrics by assuming the potential is homogeneous in a radial direction, thereby reducing the problem to an equation on the surface $S^2$ minus 3 points. We give two constructions in [50, 51]: For any cubic differential $U$ with three poles of order $\le 3$ on $\mathbb{CP}^1$, a version of Ţiţeica's equation (for an appropriate background metric)

$$\Delta\eta + 4\|U\|^2 e^{-2\eta} - 2e^{\eta} - 2\kappa = 0$$

can be solved to produce a hyperbolic affine sphere structure on $\mathbb{CP}^1$ minus the pole set [54]. Also, for a cubic differential $U$ with three poles of order 2 on $\mathbb{CP}^1$, we can solve the corresponding equation for an elliptic affine sphere

$$\Delta\eta + 4\|U\|^2 e^{-2\eta} + 2e^{\eta} - 2\kappa = 0$$

as long as $U$ is nonzero and small [51]. Then a result of Baues-Cortés [6] produces a semi-flat Calabi-Yau structure on a ball in $\mathbb{R}^3$ minus the "Y" vertex of a graph. There is also a construction of Zharkov

[82] in which the holonomy is determined and an affine Kähler metric produced, but the Monge-Ampère equation is not satisfied.

Computing the affine holonomy of such solutions is still open: This amounts to computing the projective holonomy of the solutions on $S^2$ minus the poles of $U$. The conjugacy class of the holonomy around free loops around each puncture can generally be calculated, but the problem for more complicated paths seems much harder. See the discussion above in Section 4 on surfaces and 8 on projective structures. Perhaps a place to start in terms of more global holonomy calculations is the following: Consider $U$ a meromorphic cubic differential with three poles of order 2 on $\mathbb{CP}^1$. Then, by an observation of Robert Bryant, the parabolic affine sphere metric is given by

$$\frac{|U|^2}{m^2},$$

where $m$ is the complete hyperbolic metric on $\mathbb{CP}^1$ minus the pole set of $U$. Since both the metric and cubic differential can be made reasonably explicit in this case, it is perhaps tractable to find the affine holonomy and developing map for the affine structure on $\mathbb{CP}^1$ minus 3 points determined by this parabolic affine sphere structure. The Weierstrass formula for parabolic affine spheres should help.

Compact Kähler affine manifolds were studied by Cheng-Yau in [14], and also by Shima—see e.g. [65] and [66]. Cheng-Yau's work on the analogs of Kähler-Einstein metrics on affine manifolds are strongly related to affine spheres, and so we focus on [14].

We have the following theorem of Cheng-Yau [14]

**Theorem 6.** *Let $M$ be a compact Kähler affine manifold which admits a covariant constant volume form $\nu$. Then for every smooth affine Kähler metric $g$, there is a positive constant $c$ and a smooth function $u$ so that*

$$\det\left(g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j}\right) = c\,\nu^2, \qquad g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j} > 0.$$

*In other words, the tensor $g + \nabla du$ is a Riemannian metric whose volume form is $\sqrt{c}\,\nu$.*

Note this theorem produces a Calabi-Yau metric on the tangent bundle $M^{\mathbb{C}}$, and the proof uses in an essential way Yau's estimates for producing Kähler-Einstein metrics [81].

*Remark.* Cheng-Yau actually prove a more general result that given any volume form $V$ on a compact special affine Kähler manifold, there are a constant $c$ and a function $u$ so that $g + \nabla du > 0$ and $\det(g_{ij} + u_{ij}) =$

$cV^2$. Later, Delanoë proved an analogous theorem on any compact affine Kähler manifold which does not necessarily admit a parallel volume form [25].

**Corollary 11.** *(Cheng-Yau [14]) Every compact affine Kähler manifold which admits an invariant volume form also admits a flat affine Kähler metric.*

The corollary follows from the theorem of Calabi [10] that the cubic form on an affine-complete parabolic affine sphere must vanish. This implies the affine metric is flat.

For any volume form $V$ on an affine manifold $M$, $\nabla d \log V$ is a symmetric $(0,2)$ tensor. The analogous statement for complex manifolds is that $\partial\bar{\partial}\log$ of any volume form is a $(1,1)$ form. Cheng-Yau [14] prove the following analog of Aubin and Yau's theorem on Kähler-Einstein metrics with negative Ricci curvature:

**Theorem 7.** *If $M$ is a compact affine manifold so that $\nabla d \log V > 0$, then there exists a volume form $\tilde{V}$ on $M$ so that*

$$\det\left(\frac{\partial^2 \log \tilde{V}}{\partial x^i \partial x^j}\right) = \tilde{V}^2.$$

The resulting metric $\nabla d \log V$ is the restriction of a Kähler-Einstein metric of negative Ricci curvature on $M^{\mathbb{C}}$, and thus may be called affine Kähler-Einstein metrics or Cheng-Yau metrics.

In this case, results of Koszul [39] and Vey [77] show that $M$ is an affine quotient of a convex cone containing no lines (since the closed one-form $\alpha = d\log V$ satisfies $\nabla\alpha > 0$). Cheng-Yau also prove in [14]

**Theorem 8.** *On each convex cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ which contains no lines. Then there is a unique solution to*

$$\det u_{ij} = e^{2u}, \quad u = \infty \text{ on } \partial\mathcal{C}, \quad [u_{ij}] > 0.$$

*The resulting metric $u_{ij}\, dx^i dx^j$ is the complete affine Kähler metric generating the complete Kähler-Einstein metric on the tube domain $\mathcal{C} + \sqrt{-1}\mathbb{R}^{n+1}$.*

These results of Cheng-Yau are also related to affine spheres due to a result of Sasaki [62] (see also [53]). First of all, the result of Koszul and the invariance of Kähler-Einstein metrics shows that there are affine coordinates on $M$ so that $\tilde{V}$ is homogeneous of degree $-n-1$. In these coordinates, Sasaki shows the level sets of $\tilde{V}$ are hyperbolic affine spheres.

The result of [62] shows that two of the invariant structures on convex cones determined by Cheng-Yau are equivalent: In terms of the natural affine coordinates on $\mathcal{C}$ for which the vertex of the cone is the origin, the level sets of the volume of Cheng-Yau's affine Kähler-Einstein metric are hyperbolic affine spheres. Moreover, under the foliation of the cone by homothetic copies of the hyperbolic affine sphere, the affine Kähler-Einstein metric is a metric product of the affine metric on the hyperbolic affine sphere and the flat metric $dr^2/r^2$ on the radial parameter in $\mathbb{R}^+$. Cheng-Yau's affine Kähler-Einstein metric is invariant under any linear automorphism of the cone, and so it descends to any affine quotient manifold.

There are other invariant affine Kähler metrics on regular convex cones. The *characteristic function* on $\mathcal{C} \subset \mathbb{R}^n$ is defined by

$$\psi(x) = \int_{\mathcal{C}^*} e^{-\langle x, x^* \rangle} dx^*$$

for $\mathcal{C}^*$ the dual cone of $\mathcal{C}$. Then Koszul-Vinberg's metric $\nabla d \log \psi$ is an affine Kähler metric on $\Omega$ invariant under linear automorphisms of $\mathcal{C}$. Thus this metric too descends to any affine manifold which is a quotient of $\mathcal{C}$ by a group of linear automorphisms acting discretely and properly discontinuously. One can think of Koszul-Vinberg's metric as an analog of the Bergman metric in complex geometry, but this metric is not, for general non-homogeneous cones, the restriction of the Bergman metric on the tube domain $\Omega + \sqrt{-1}\mathbb{R}^n$ (see e.g. [62] for an investigation of these metrics).

The level sets of the volume form of these affine Kähler metrics are also invariant hypersurfaces asymptotic to the boundary of the convex cone $\mathcal{C}$, though they do not in general seem to have the same duality property under the conormal map as the hyperbolic affine sphere. See e.g. Darvishzadeh-Goldman [23] for applications to convex $\mathbb{RP}^2$ surfaces.

## 10. Affine maximal hypersurfaces

A natural generalization of the concept of a parabolic affine sphere is an affine maximal hypersurface. The condition for a parabolic affine sphere is that the affine shape operator $S$ vanishes, while a hypersurface is affine maximal if the affine mean curvature $H = \frac{1}{n}\mathrm{tr}\, S = 0$.

The volume induced by affine metric provides a natural equiaffine-invariant functional on all smooth strictly convex hypersurfaces. The first variation of this functional is

$$H = 0$$

for $H = \frac{1}{n}\mathrm{tr} S$ the affine mean curvature. By analogy with the Riemannian case, Blaschke then referred to these stationary hypersurfaces as affine minimal [7]. Much later, Calabi found that the second variation is negative in many cases. Therefore, such a stationary hypersurface is a local maximum for the volume functional, and thus we now call stationary *affine maximal hypersurfaces.* Calabi proved [11]

**Theorem 9.** *Let $L$ be an affine maximal hypersurface in $\mathbb{R}^{n+1}$. Then the second variation under any compactly supported interior deformation is negative if either*

- $n = 2$.
- *For any $n$, if $L$ is locally a graph of the form $x^{n+1} = f(x^1, \dots, x^n)$. Moreover, $L$ is a global maximum among such variations in this case.*

For $\Omega \subset \mathbb{R}^n$ a domain and a $u \colon \Omega \to \mathbb{R}$ a smooth convex function, the graph $(x, u(x))$ is an affine maximal hypersurface if and only if the fourth-order PDE

$$U^{ij} D_{ij}[(\det u_{ij})^{-\frac{n+1}{n+2}}] = 0$$

is satisfied, where $[U^{ij}]$ is the cofactor matrix of the Hessian matrix $[u_{ij}]$.

Chern conjectured that any properly embedded affine maximal surface in $\mathbb{R}^3$ must be an elliptic paraboloid [16]. This extension of Jörgens's Theorem was proved by Trudinger-Wang [72]. The corresponding problem is open in higher dimension, but non-smooth viscosity solutions in dimension $n \geq 10$ are presented in [72].

**Theorem 10.** [72] *For $\Omega$ a domain in $\mathbb{R}^2$, and $u \colon \Omega \to \mathbb{R}$ is a smooth convex function whose graph $(x, u(x))$ is a properly embedded affine maximal surface in $\mathbb{R}^3$, then $u$ is a quadratic function.*

In order to prove this theorem, Trudinger-Wang use the estimates of Caffarelli-Gutiérrez [8] on solutions of the linearized Monge-Ampère equation. Earlier, Calabi had proved that any affine maximal surface in $\mathbb{R}^3$ which is both Euclidean and affine complete must be an elliptic paraboloid [12]. In fact, affine completeness implies Euclidean completeness for hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geq 2$ by [73], and so we have the following result, also proved by Jia-Li [45]:

**Theorem 11.** *Any affine-complete maximal surface in $\mathbb{R}^3$ is an elliptic paraboloid.*

We also remark that affine maximal surfaces are important examples in the theory of integrable systems, with Chern-Terng's construction of Bäcklund transformations for them [17]. The Weierstrass formula for affine maximal surfaces is given by Calabi [12], Terng [70], and Li [42].

Each affine maximal surface is given by a holomorphic curve $Z \to \mathbb{C}^3$, and the parametrization may be recovered by

$$-\sqrt{-1}\left(Z \times \bar{Z} + \int Z \times dZ - \int \bar{Z} \times d\bar{Z}\right)$$

for $\times$ the cross product.

## 11. Affine normal flow

Affine spheres are the solitons (self-similar solutions) to the affine normal flow. If a hypersurface $L$ is parametrized locally by an immersion $f$, the affine normal flow is

$$\frac{\partial f}{\partial t} = \xi$$

for $\xi$ the affine normal. Hyperbolic, parabolic, and elliptic affine spheres are then respectively expanding, translating, and contracting soliton solutions for the affine normal flow. Even though the affine normal $\xi$ is a third-order invariant of $f$, the affine normal flow is equivalent to the second-order parabolic flow of the $\frac{1}{n+2}$ power of the Gauss curvature (since $\xi = K^{\frac{1}{n+2}}\nu$ plus a tangential part, for $\nu$ the Euclidean unit normal).

Chow [19] shows that compact smooth strictly convex initial hypersurfaces in $\mathbb{R}^{n+1}$ converge in finite time under the affine normal flow, and Andrews [3] proves that the rescaled limit of such contracting solutions is an ellipsoid. Andrews [4] also shows that arbitrary compact convex initial hypersurfaces are instantaneously regularized under the affine normal flow.

Recently, Tsui and the author [49] extended the affine normal flow to noncompact convex initial hypersurfaces. One of the consequences is a parabolic proof of Cheng-Yau's theorem on the existence of hyperbolic affine spheres:

**Theorem 12.** *Given any open convex cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ which contains no lines, the affine normal flow evolves the initial hypersurface $\partial C$ to homothetically expanding copies of the hyperbolic affine sphere asymptotic to $\partial C$.*

This theorem recovers Cheng-Yau's solution to the Monge-Ampère equation Dirichlet problem

$$\det u_{ij} = \left(-\frac{1}{u}\right)^{n+2}, \qquad u|_{\partial\Omega} = 0$$

for $\Omega \subset \mathbb{R}^n$ any convex bounded domain[13], together with the result of Gigena [30] and Sasaki [61] that the solution to such an equation is a hyperbolic sphere asymptotic to the cone over $\Omega$. In [49, 48], we also classify all ancient solutions to the affine normal flow, showing them to be either ellipsoids or paraboloids.

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