AFFINE MANIFOLDS, SYZ GEOMETRY AND THE "Y" VERTEX

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ABSTRACT. We study the real Monge-Ampère equation in two and three dimensions, both from the point of view of the SYZ conjecture, where solutions give rise to semi-flat Calabi-Yau's and in affine differential geometry, where solutions yield parabolic affine sphere hypersurfaces. We find explicit examples, connect the holomorphic function representation to Hitchin's description of special Lagrangian moduli space, and construct the developing map explicitly for a singularity corresponding to the type I_n elliptic fiber (after hyper-Kähler rotation). Following Baues and Cortés, we show that various types of metric cones over two-dimensional elliptic affine spheres generate solutions of the Monge-Ampère equation in three dimensions. We then prove a local and global existence theorem for an elliptic affine two-sphere metric with prescribed singularities. The metric cone over the two-sphere minus three points yields a parabolic affine sphere with singularities along a "Y"-shaped locus. This gives a semi-flat Calabi-Yau metric in a neighborhood of the "Y" vertex.

1. INTRODUCTION

The basic question we would like to understand is, What does the geometry of a Calabi-Yau manifold look like near (or "at") the large complex structure limit point? In order to answer this question, one first fixes the ambiguity of rescaling the metric by an overall constant. Gromov proved that Ricci-flat manifolds with fixed diameter have a limit under the Gromov-Hausdorff metric (on the space of metric spaces).

Now by the conjecture of [25], one expects that near the limit, the Calabi-Yau has a fibration by special Lagrangian submanifolds which are getting smaller and smaller (than the base). The reason can be found by looking at the mirror large radius limit. Fibers are mirror to the zero brane, and the base is mirror to the $2n$ brane, which becomes large at large radius. Metrically, the Calabi-Yau geometry should be roughly a fibration over the moduli space of special Lagrangian tori (T). The dual fibration is by dual tori, Hom $(\pi_1(T), \mathbb{R})$/Hom $(\pi_1(T), \mathbb{Z})$. The flat fiber geometry of the dual torus fibration has a flat fiber dual, which is not the same as the original geometry, but should be the same after corrections by disk instantons. These should get small in the limit of small tori, though. Namely, we expect that the Gromov-Hausdorff limit of a fixed-diameter Calabi-Yau manifold approaching a maximal-degeneration point carries the same geometry as the moduli space of special Lagrangian tori. That is, it is a manifold (and an affine manifold at that) of half the dimension.

Further, the Calabi-Yau near the limit should be "asymptotically close" to the standard flat torus fibration over special Lagrangian tori moduli space, whose fibers are the flat tori (we get this from dual torus considerations applied to the mirror manifold). This space is a quotient of the tangent space of the moduli space (which is Hessian), and the Calabi-Yau condition means that the limiting affine manifold metric should be Monge-Ampère ($\det(\operatorname{Hess}\Phi) = 1$). Global considerations require that the lattice defining the torus (generated by the vectors associated to the Hessian coordinates) is well-defined, meaning that the Monge-Ampère manifold has affine transition functions in the semi-direct product of $SL(n, \mathbb{Z})$ with $\mathbb{R}^n$ translations.

The above is an informal (read: "physical") explanation behind the conjecture made independently by Gross-Wilson [13] and Kontsevich-Soibelman [16] (it appears in the work of Fukaya [9] as well), and proved by Gross-Wilson for the special case of K3 surfaces [13]. Their proof uses the Ooguri-Vafa [21] metric in the neighborhood of a torus degeneration to build an approximate Ricci-flat metric on the entirety of an elliptic K3 with 24 singular fibers (with an elliptic-fibration/stringy-cosmic-string metric outside the patches describing degenerations).

Another aspect of the conjecture is that the limiting manifold has singularities in codimension two, with monodromy transformations defined for each loop about the singular set. Gross has shown the existence of a limiting singular set of codimension two for (non-Lagrangian) torus fibrations on toric three-fold Calabi-Yau's, and further has shown that the limiting singular set has the structure of a trivalent graph. Taking our cue from this work, then in three dimensions a point on the limiting manifold may be a smooth point, a point near an interval singularity, or the trivalent vertex of a "Y"-shaped singularity locus (these vertices have a subclassification based on the monodromies near the vertex). Examples of explicit Monge-Ampère metrics for points of the first two types are known, the interval singularity reducing to the an interval times the two-fold point singularity. The absence of a local metric model of a trivalent vertex singularity limits our ability to prove this conjecture in three dimensions. Even if we had such a model, it might not suffice to prove the conjectures about the limiting metric, just as the non-Ooguri-Vafa elliptic fibration metric does not suffice even in the two-dimensional case. Still, we regard the existence of a semi-flat Calabi-Yau metric near the "Y" vertex as an important first step in addressing these conjectures in three dimensions.[1]

We therefore concern ourselves with studying Monge-Ampère manifolds in low dimensions, with the goal of finding a local model for a trivalent degeneration of special Lagrangian tori. Taking our model to be a metric cone over a thrice-punctured two-sphere, we have, by an argument of Baues and Cortés, that the sphere metric should be an elliptic affine sphere on S^2 with three singularities. The singularity type of the metric at the three points is fixed by the pole behavior of a holomorphic cubic form. Our main result is a proof of the existence of such an elliptic affine sphere, hence its cone, which solves the Monge-Ampère equation with the desired singular locus.

The plan of attack is as follows. We study the Monge-Ampère equation and Hessian manifolds in Section 2, exhibiting a few new solutions in three-dimensions. In Section 3, we study the relation between elliptic fibrations ("stringy cosmic string") and Hessian coordinates in two dimensions. In Section 4, we construct two-dimensional Monge-Ampère solutions as parabolic affine two-spheres in $\mathbb{R}^3$, using affine differential geometry techniques developed by Simon and Wang. We find the affine coordinates and calculate the monodromy for some of our solutions in Section 5. In Section 6 we discuss parabolic affine spheres which occur as radial ("cone") metrics, generalizing Baues and Cortés's result to some new examples. Simon and Wang's techniques also give equations for singular elliptic affine spheres, the cone over which yields three-dimensional Monge-Ampére solutions. In Section 7 we employ this technique and study the local structure near a singularity of the elliptic affine sphere. Using this local analysis, we turn to global case and complete the existence proof. The Calabi-Yau metric near the "Y" vertex is then constructed as a cone.

2. Hessian Metrics and the Monge-Ampère Equation

We recall that a metric is of Hessian type if in coordinates $\{x^i\}$ it has the form $ds^2 = \Phi_{ij} dx^i \otimes dx^j$, where $\Phi_{ij} = \partial^2\Phi/\partial x^i \partial x^j$. Hitchin proved [14] that natural metric ("McLean" or "Weil-Petersson") on moduli space of special Lagrangian submanifolds naturally has this

[1] This vertex is not the same as the topological vertex of [1], which appears at a corner of the toric polyhedron describing the Calabi-Yau. The relation between the toric description of the Calabi-Yau and the singularities of the special Lagrangian torus fibration has been discussed in [12].

structure, and the semi-flat metric on the complexification (by flat bundles) defined by the Kähler potential Φ is Ricci flat if

$$\det\,(\Phi_{ij}) = 1. \tag{1}$$

In Hessian coordinates we can compute the Christoffel symbols $\Gamma^i{}_{jk} = \frac{1}{2}\Phi^{il}\Phi_{jkl}$ (where $\nabla_j\partial_k = \Gamma^i{}_{jk}\partial_i$), and defining the curvature tensor $R_{ij}{}^k{}_l = \partial_i\Gamma^k{}_{jl} + \Gamma^k{}_{im}\Gamma^m{}_{jl} - (i \leftrightarrow j)$ by $[\nabla_i, \nabla_j]\partial_k = R_{ij}{}^k{}_l\partial_l$, we find

$$R_{ijkl} = -\frac{1}{4}\Phi^{ab}[\Phi_{ika}\Phi_{jlb} - \Phi_{jka}\Phi_{ilb}]. \tag{2}$$

2.1. **Hessian Coordinate Transformations.** One asks, what coordinate transformations preserve the Hessian form of the metric? If we try to write $ds^2 = \Phi_{ij}dx^i dx^j = \Psi_{ab}dy^a dy^b = \Psi_{ab}y^a{}_i y^b{}_j dx^i dx^j$, then the consistency equations $\Phi_{ijk} = \Phi_{kji}$ yield conditions on the coordinate transformation $y(x)$. Specifically, we have $\partial_k(\Psi_{ab}y^a{}_i y^b{}_j) = \partial_i(\Psi_{ab}y^a{}_k y^b{}_j)$, which is equivalent to

$$\Psi_{ab}(y^a{}_i y^b_{jk} - y^a{}_k y^b{}_{ij}) = 0.$$

In two dimensions, for example, there can be many solutions to these equations. In Euclidean space $\Psi_{ab} = \delta_{ab}$ with coordinates y^a, if we put $y^1 = f(x^1 + x^2) + g(x^1 - x^2)$ and $y^2 = f(x^1 + x^2) - g(x^1 - x^2)$, then the equations are solved and we can find $\Phi(x)$. For example, if $f(s) = g(s) = s^2/2$, we find $\Phi(x) = [(x^1)^4 + 6x^1x^2 + (x^2)^4]/12$.

Note that this transformation is not affine. Thus Hessian metrics may exist on non-affine manifolds. Hessian manifolds can be characterized as locally having an abelian Lie algebra of gradient vector fields acting simply transitively [23]. Though Hessian manifolds are more general than affine manifolds, a Hessian manifold appearing as a moduli space of special Lagrangian tori must have an affine structure. We therefore focus on affine Hessian manifolds in this paper.

2.2. **Examples of Monge-Ampère Metrics.** As we will see in Section 3, there are many Monge-Ampère metrics in two dimensions, but a paucity of examples in three or more dimensions. Here we provide a few solutions.

Example 1. *In dimension d consider the ansatz $\Phi = \Phi(r)$, where $r = \sqrt{\sum_i (x^i)^2}$. As shown by Calabi, the equation (1) is solved if*

$$\Phi(r) = \int (1 + r^d)^{1/d}. \tag{3}$$

The rescalings $r \to cr$ and $\Phi \to c\Phi$ also have constant $\det(\mathrm{Hess}\,\Phi)$. *For example, in two dimensions ($d = 2$), $\Phi(r) = \sinh^{-1}(r) + r\sqrt{1 + r^2}$ is a solution.*

Example 2. *We now take* $d = 3$ *and make the axial ansatz* $\Phi = \Phi(\rho, z)$, *where* $\rho = \sqrt{x^2 + y^2}$. *One computes* $\det(\text{Hess }\Phi) = \frac{\Phi_\rho}{\rho}[\Phi_{zz}\Phi_{\rho\rho} - (\Phi_{\rho z})^2]$. *We search for a solution of the form* $\Phi(\rho, z) = A(\rho)B(z)$, *which leads to the equation*

$$AA'A''B^2B'' - (A')^3B(B')^2 = \rho.$$

We take $A = \frac{3}{4}\rho^{4/3}$, *so* $(A')^3 = \rho$ *and* $AA'A'' = \rho/4$. *This gives the equation* $B^2B'' - 4B(B')^2 = 4$. *The substitution* $B = -2^{1/3}v^{-1/3}$ *leads to the equation* $v'' = 6v^2$. *This equation is satisfied by the Weierstrass* $\mathfrak{p}$*-function corresponding to a Weierstrass elliptic curve with* $g_2 = 0$ *and* g_3, *a constant of integration, real (so that* $\mathfrak{p}$ *is real). The full solution is then*

$$\Phi(\rho, z) = -3 \cdot 2^{-5/3}\rho^{4/3}\mathfrak{p}_\tau(z + c)^{-1/3},$$

where $\tau = \tau(g_2 = 0, g_3 < 0)$ *and* c *is an arbitrary constant. In order for this function to define a metric, its Hessian matrix must be positive definite. One checks that all diagonal minors are positive as long as* $\mathfrak{p} < 0$ *and* $g_3 < 0$ *(hence the condition above).*

Example 3. *In three dimensions, when* $\Phi = \Phi(t)$ *for some function* $t(x_1, x_2, x_3)$, *there is sometimes simplification of the Monge-Ampère equation – for example,* $t = r$ *leads to the simple solution (3). For* $\Phi = \Phi(t(x))$, *we compute:*

$$\begin{aligned} det(\Phi_{ij}) &= (\Phi')^3 det(t_{ij}) + \\ &\quad (\Phi')^2(\Phi'')\left[t_1^2(t_{22}t_{33} - t_{23}^2) + t_2^2(t_{33}t_{11} - t_{13}^2) + \right. \\ &\quad t_3^2(t_{11}t_{22} - t_{12}^2) + 2t_1t_2(t_{13}t_{23} - t_{12}t_{33}) + \\ &\quad \left. 2t_2t_3(t_{12}t_{13} - t_{11}t_{23}) + 2t_1t_3(t_{12}t_{23} - t_{22}t_{13})\right]. \end{aligned}$$

We reduce to an ODE when $det(t_{ij})$ *and the term in brackets can be written in terms of* t *alone.*

For example, if $t = xyz$ *we find* $det(t_{ij}) = 2t$, *and the term in brackets is* $3t^2$. *This leads to* $(\Phi')^3(3t^2) + (\Phi')^2(\Phi'')(2t) = 1$, *which is solved by putting* $y = (\Phi')^3$. *Then* $(yt^2)' = 1$, *which leads to* $\Phi = \int(Ct^{-2} + t^{-1})^{1/3}dt$. *One calculates that* $Hess(\Phi)$ *is positive definite if* $t < 0$ *and* $|C| > |t|$.

Example 4. *For another example of the type in Ex. 3, we put* $t = xy + yz + zx$, *then* $det(t_{ij}) = 2$ *and the term in brackets is* $4t$. *We find* $\Phi = \int(1/2 + Ct^{-3/2})^{1/3}dt$. *This solution is also convex in a* C*-dependent region.*

3. Monge-Ampère Metrics and the Stringy Cosmic String

Using a hyper-Kähler rotation we can treat any elliptic surface as a special Lagrangian fibration and try to find its associated Hessian coordinates and – if the fibers are flat – the corresponding solution to the Monge-Ampère equation.

In the case of the stringy cosmic string, we begin with a semi-flat fibration with torus fiber coordinates $t \sim t+1$ and $x \sim x+1$. As a holomorphic fibration, the stringy cosmic string is defined by a holomorphic modulus $\tau(z)$. One can derive the Kähler potential through the Gibbons-Hawking ansatz (with $\partial/\partial t$ as Killing vector) using connection one-form $A = -\tau_1 dx$ and potential $V = \tau_2$ (so $*dA = dV$), then solving for the holomorphic coordinate. One finds $\xi = t + \tau(z)x = t + \tau_1 x + i\tau_2 x$. The hyper-Kähler structure is specified by the forms

$$\begin{aligned} \omega_1 &= dx \wedge dt + (\frac{i}{2})\tau_2 dz \wedge d\overline{z} \\ \omega_2 + i\omega_3 &= dz \wedge d\xi. \end{aligned}$$

The stringy cosmic string solution starts directly from the Kähler potential $K(z,\xi) = \xi_2/\tau_2 + k(z,\overline{z})$, where $\partial_z \partial_{\overline{z}} k = \tau_2$.

We seek the Hessian coordinates for the base of the semi-flat special Lagranian torus fibration. In coordinates (x, t, z_1, z_2) the metric has the block diagonal form

$$Q \oplus R \equiv \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} \oplus \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}. \qquad (4)$$

For a semi-flat fibration over a Hessian manifold in Hessian coordinates, the base-dependent metric on the fiber looks the same as the metric on the base. Therefore, we would need to find coordinates $u_1(z,\overline{z}), u_2(z,\overline{z})$ so that the metric in u-space looks like Q in (4). This is accomplished if the change of basis matrix $M_{ij} = \partial z_i/\partial u_j$ obeys $M^T M = Q/\tau_2$. A calculation reveals the general solution to be $M = O\widetilde{M}$, where $\widetilde{M} = \frac{1}{\tau_2}\begin{pmatrix} \tau_2 & \tau_1 \\ 0 & 1 \end{pmatrix}$, and O is an orthogonal matrix. The same result can be obtained using Hitchin's method, which we now review.

Hitchin [14] obtains Hessian coordinates on the moduli space of special Lagrangian submanifolds from period integrals. In order to apply this technique here we first make a hyper-Kähler rotation, so that the fibration is special Lagrangian, by putting $\omega = \omega_2$, $\mathrm{Im}\Omega = \omega_3$. Explicitly, for each base coordinate z_i we construct closed one forms on the Lagrangian L, defined by $\iota_{\partial/\partial z_i}\omega = \theta_i$ and compute the periods $\lambda_{ij} = \int_{A_i} \theta_j$, where $\{A_i\}$ is a basis for $H_1(L,\mathbb{Z})$. In our case we readily find $\theta_1 = dt + \tau_1 dx$, $\theta_2 = -\tau_2 dx$, and, using the basis $A_1 = \{t \to t+1\}$, $A_2 = -\{x \to x+1\}$, we get

$$\lambda_{ij} = \begin{pmatrix} 1 & 0 \\ -\tau_1 & \tau_2 \end{pmatrix}.$$

The forms $\lambda_{ij} dz_j$ are closed on the base and we set them equal to du_i. This defines the coordinates du_i up to constants, and we find $u_1 = z_1$, $u_2 = -\text{Re}\phi$, where $\partial_z \phi = \tau$. To connect with the solution above, one easily inverts the matrix $\lambda_{ij} = \partial u_i / \partial z_j$ to find the matrix $\partial z_i / \partial u_j = \widetilde{M}$.

Remark. The change of coordinates $u_2 \leftrightarrow -u_2$ preserves the Monge-Ampère equation, and we sometimes use the latter in what follows.

Legendre dual coordinates v_i are defined as follows. Define $(d-1)$-forms ψ_i (here $d = 2$ so the ψ's are also one-forms) by putting $\Omega = dz_i \wedge \psi_i$ and compute the periods $\mu_{ij} = \int_{B_i} \psi_j$, where $B_j \in H_{d-1}(L, \mathbb{Z})$ are Poincaré dual to the A_i. In our example, $\psi_1 = \tau_2 dx$, $\psi_2 = dt + \tau_1 dx$, $B_1 = \{x \to x+1\}$, $B_2 = \{t \to t+1\}$, and

$$\mu_{ij} = \begin{pmatrix} \tau_2 & \tau_1 \\ 0 & 1 \end{pmatrix}.$$

(Note $\lambda^T \mu$ is symmetric, as required.) Setting $dv_i = \mu_{ij} dz_j$ we find $v_1 = \text{Im}\phi$ and $v_2 = z_2$.

Hitchin showed that the coordinates u_i and v_i are related by the Legendre transformation defined by the function Φ whose Hessian gives the metric. Namely, $v_i = \partial\Phi/\partial u_i$. We can think of Φ as a function of the $z_i(u_j)$ and differentiate with respect to u_j using the chain rule. (We find $\partial z_i/\partial u_j$ by inverting the matrix of derivatives $\partial u_i/\partial z_j$.) One finds

$$\Phi_{z_1} = \text{Im}\,\phi - \tau_1 z_2, \qquad \Phi_{z_2} = \tau_2 z_2.$$

(For the transformed potential Ψ we have $\Psi_{z_1} = \tau_2 z_1$ and $\Psi_{z_2} = \tau_1 z_1 - \text{Re}\phi$.) The solution can be given in terms of another holomorphic antiderivative,[2] χ, such that $\partial_z \chi = \phi$.

$$\Phi = -z_2 \text{Re}\phi + \text{Im}\chi.$$

Note, then, that being able to write down the explicit Hessian potential depends only on our ability to integrate τ and invert the functions $u_i(z_j)$. The Legendre-transformed potential is $\Psi = z_1 \text{Im}\phi - \text{Im}\chi$.

[2] V. Cortés has found a generalization of this potential as the defining function of parabolic affine hyperspheres – equivalently, Hessian manifolds obeying the Monge-Ampère equation – described by holomorphic data. The example above is the simplest instance of his more general approach, and is due to Blaschke.

Example 5. $\tau = 1/z$. *If we put* $z = re^{i\theta}$ *and take* $\tau = 1/z$ *then* $\phi = \log z$, *so* $u_1 = z_1$ *and* $u_2 = -\mathrm{Re}\phi = -\log r$. *Thus* $\Phi(u_1, u_2) = -z_2 \log\sqrt{z_1^2 + z_2^2} + \int \log\sqrt{z_1^2 + z_2^2} dz_2$, *where* $z_1 = u_1$ *and* $z_2 = \sqrt{e^{-2u_2} - u_1^2}$. *Since* $v_1 = \mathrm{Im}\Phi = \tan^{-1}(z_2/z_1)$ *and* $v_2 = z_2$, *we may solve the equations* $\partial\Phi/\partial u_i = v_i$ *to find*

$$\Phi = u_1 \left[\tan^{-1}\left(\sqrt{(e^{-u_2}/u_1)^2 - 1}\right) - \sqrt{(e^{-u_2}/u_1)^2 - 1}\right].$$

One easily checks that $\det(\Phi_{ij}) = 1$.

To summarize, let z be a holomorphic coordinate on the base of a semi-flat elliptic fibration. Let $\tau = \tau(z)$ be the holomorphically varying modulus of the elliptic curve on the fiber. Then we define ϕ, χ holomorphic so that

$$\phi_z = \tau, \qquad \chi_z = \phi.$$

Let $z = z_1 + iz_2$ represent real and imaginary parts, with similar notation for the real and imaginary parts of τ, ϕ, χ. Then affine flat coordinates u_1, u_2 may be chosen as

$$u_1 = z_1, \qquad u_2 = -\phi_1.$$

The metric on the base is given by

$$\tau_2 |dz|^2 = \frac{\partial^2 \Phi}{\partial u_i \partial u_j} du_i du_j$$

for the affine Kähler potential Φ, which satisfies

$$\Phi = -z_2\phi_1 + \chi_2.$$

The Legendre dual coordinates $v_i = \partial\Phi/\partial u_i$ are given by

$$v_1 = \phi_2, \qquad v_2 = -z_2.$$

The potential Ψ in the v coordinates is the Legendre transform of Φ:

$$\Psi = u_1 v_1 + u_2 v_2 - \Phi = z_1\phi_2 - \chi_2.$$

Φ and Ψ satisfy the Monge-Ampère equation

$$\det\left(\frac{\partial^2\Phi}{\partial u_i \partial u_j}\right) = 1, \qquad \det\left(\frac{\partial^2\Psi}{\partial v_i \partial v_j}\right) = 1.$$

The metric satisfies

$$\tau_2|dz|^2 = \frac{\partial^2\Phi}{\partial u_i\partial u_j} du_i du_j = \frac{\partial^2\Psi}{\partial v_i \partial v_j} dv_i dv_j.$$

4. Explicit Solutions for the Developing Map

This is a model of a 2-dimensional parabolic affine sphere. At first we work from the structure equations, to find an ODE problem to solve. We find a family of solutions indexed by an integer k. The $k = 0$ case has monodromy, and is found to be equivalent to the holomorphic representation seen earlier. with $\tau = 2i \log z$. The ODE approach is more difficult, but is probably all that's available in the elliptic affine sphere case. Moreover, the ODE approach has been used successfully to calculate monodromy for hyperbolic affine spheres [19], and also for a global existence result for parabolic affine spheres on S^2 minus singular points [20]. (Recall we are interested in elliptic affine spheres since a cone over an elliptic affine sphere in dimension two is a parabolic affine sphere in dimension three.)

4.1. **Simon and Wang's developing map.** U. Simon and C.P. Wang [24] formulate the condition for a two-dimensional surface to be an affine sphere in terms of the conformal geometry given by the affine metric. Since we rely heavily on this work, we give a version of the arguments here for the reader's convenience. For basic background on affine differential geometry, see Calabi [6], Cheng-Yau [7] and Nomizu-Sasaki [22].

Consider a 2-dimensional parabolic affine sphere in $\mathbb{R}^3$. Then the affine metric gives a conformal structure, and we choose a local conformal coordinate $z = x + iy$ on the hypersurface. Then the affine metric is given by $h = e^{\psi}|dz|^2$ for some function ψ. Parametrize the surface by $f : \mathcal{D} \to \mathbb{R}^3$, with $\mathcal{D}$ a domain in $\mathbb{C}$. Since $\{e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y\}$ is an orthonormal basis for the tangent space, the affine normal ξ must satisfy this volume condition (see e.g. [22])

$$\det(e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y, \xi) = 1, \tag{5}$$

which implies

$$\det(f_z, f_{\bar z}, \xi) = \tfrac{1}{2} i e^{\psi}. \tag{6}$$

Now only consider parabolic affine spheres. In this case, the affine normal ξ is a constant vector, and we have

$$\begin{cases} D_X Y = \nabla_X Y + h(X,Y)\xi \\ \qquad D_X \xi = 0 \end{cases} \tag{7}$$

Here D is the canonical flat connection on $\mathbb{R}^3$, ∇ is a flat connection, and h is the affine metric.

It is convenient to work with complexified tangent vectors, and we extend ∇, h and D by complex linearity. Consider the frame for the tangent bundle to the surface $\{e_1 = f_z = f_*(\frac{\partial}{\partial z}), e_{\bar 1} = f_{\bar z} = f_*(\frac{\partial}{\partial \bar z})\}$. Then we have

$$h(f_z, f_z) = h(f_{\bar z}, f_{\bar z}) = 0, \quad h(f_z, f_{\bar z}) = \tfrac{1}{2}e^{\psi}. \tag{8}$$

Consider θ the matrix of connection one-forms

$$\nabla e_i = \theta_i^j e_j, \quad i, j \in \{1, \bar 1\},$$

and $\hat\theta$ the matrix of connection one-forms for the Levi-Civita connection. By (8)

$$\hat\theta_{\bar 1}^1 = \hat\theta_1^{\bar 1} = 0, \quad \hat\theta_1^1 = \partial\psi, \quad \hat\theta_{\bar 1}^{\bar 1} = \bar\partial\psi. \tag{9}$$

The difference $\hat\theta - \theta$ is given by the Pick form. We have

$$\hat\theta_i^j - \theta_i^j = C_{ik}^j \rho^k,$$

where $\{\rho^1 = dz, \rho^{\bar 1} = d\bar z\}$ is the dual frame of one-forms. Now we differentiate (6) and use the structure equations (7) to conclude

$$\theta_1^1 + \theta_{\bar 1}^{\bar 1} = d\psi.$$

This implies, together with (9), the apolarity condition

$$C_{1k}^1 + C_{\bar 1 k}^{\bar 1} = 0, \quad k \in \{1, \bar 1\}.$$

Then, when we lower the indices, the expression for the metric (8) implies that

$$C_{\bar 1 1 k} + C_{1 \bar 1 k} = 0.$$

Now C_{ijk} is totally symmetric on three indices [7, 22]. Therefore, the previous equation implies that all the components of C must vanish except C_{111} and $C_{\bar 1\bar 1\bar 1} = \overline{C_{111}}$.

This discussion completely determines θ:

$$\begin{pmatrix} \theta_1^1 & \theta_{\bar 1}^1 \\ \theta_1^{\bar 1} & \theta_{\bar 1}^{\bar 1} \end{pmatrix} = \begin{pmatrix} \partial\psi & C_{\bar 1\bar 1}^1 d\bar z \\ C_{11}^{\bar 1} dz & \bar\partial\psi \end{pmatrix} = \begin{pmatrix} \partial\psi & \bar U e^{-\psi} d\bar z \\ U e^{-\psi} dz & \bar\partial\psi \end{pmatrix}, \tag{10}$$

where we define $U = C_{11}^{\bar 1} e^{\psi}$.

Recall that D is the canonical flat connection induced from $\mathbb{R}^3$. (Thus, for example, $D_{f_z} f_z = D_{\frac{\partial}{\partial z}} f_z = f_{zz}$.) Using this statement, together with (8) and (10), the structure equations (7) become

$$\left\{ \begin{array}{r} f_{zz} = \psi_z f_z + U e^{-\psi} f_{\bar z} \\ f_{\bar z\bar z} = \bar U e^{-\psi} f_z + \psi_{\bar z} f_{\bar z} \\ f_{z\bar z} = \frac{1}{2} e^{\psi} \xi \end{array} \right. \tag{11}$$

Then, together with the equations $\xi_z = \xi_{\bar z} = 0$, these form a linear first-order system of PDEs in ξ, f_z and $f_{\bar z}$:

$$\frac{\partial}{\partial z}\begin{pmatrix}\xi\\ f_z\\ f_{\bar z}\end{pmatrix} = \begin{pmatrix}0&0&0\\ 0&\psi_z&Ue^{-\psi}\\ \frac12 e^{\psi}&0&0\end{pmatrix}\begin{pmatrix}\xi\\ f_z\\ f_{\bar z}\end{pmatrix}, \tag{12}$$

$$\frac{\partial}{\partial \bar z}\begin{pmatrix}\xi\\ f_z\\ f_{\bar z}\end{pmatrix} = \begin{pmatrix}0&0&0\\ \frac12 e^{\psi}&0&0\\ 0&\bar U e^{-\psi}&\psi_{\bar z}\end{pmatrix}\begin{pmatrix}\xi\\ f_z\\ f_{\bar z}\end{pmatrix}. \tag{13}$$

In order to have a solution of the system (11), the only condition is that the mixed partials must commute (by the Frobenius theorem). Thus we require

$$\begin{aligned}\psi_{z\bar z} + |U|^2 e^{-2\psi} &= 0,\\ U_{\bar z} &= 0.\end{aligned} \tag{14}$$

The system (11) is an initial-value problem, in that given (A) a base point z_0, (B) initial values $f(z_0) \in \mathbb{R}^3$, $f_z(z_0)$ and $f_{\bar z}(z_0) = \overline{f_z(z_0)}$, and (C) U holomorphic and ψ which satisfy (14), we have a unique solution f of (11) as long as the domain of definition $\mathcal{D}$ is simply connected. We then have that the immersion f satisfies the structure equations (7). In order for ξ to be the affine normal of $f(\mathcal{D})$, we must also have the volume condition (6), i.e. $\det(f_z, f_{\bar z}, \xi) = \frac12 i e^{\psi}$. We require this at the base point z_0 of course:

$$\det(f_z(z_0), f_{\bar z}(z_0), \xi) = \tfrac12 i e^{\psi(z_0)}. \tag{15}$$

Then use (11) to show that the derivatives with respect to z and $\bar z$ of $\det(f_z, f_{\bar z}, \xi)e^{-\psi}$ must vanish. Therefore the volume condition is satisfied everywhere, and $f(\mathcal{D})$ is a parabolic affine sphere with affine normal ξ.

Using (11), we compute

$$\det(f_z, f_{zz}, \xi) = \tfrac12 i U, \tag{16}$$

which implies that U transforms as a section of K^3, and $U_{\bar z} = 0$ means it is holomorphic.

Note that equation (14) is in local coordinates. In other words, if we choose a local conformal coordinate z, then the Pick form $\mathbf{U} = U\,dz^3$, and the metric is $h = e^{\psi}|dz|^2$. Then plug U, ψ into (14). In a patch with a new holomorphic coordinate $w(z)$, the metric will have the form $e^{\widetilde\psi}|dw|^2$, with cubic form $\widetilde U dw^3$. Then $\widetilde\psi(w), \widetilde U(w)$ will satisfy (14).

It will be useful to write the relations in real coordinates. Note that $f_z = \frac12(f_x - i f_y)$, and $f_{zz} = \frac14(f_{xx} - f_{yy} - 2i f_{xy})$, $f_{z\bar z} = \frac14(f_{xx} + f_{yy})$.

Now write $U = U_1 + iU_2$. We find

$$\begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 \\ e^\psi & \frac12(\psi_x + 2U_1e^{-\psi}) & -\frac12(\psi_y + 2U_2e^{-\psi}) \\ 0 & \frac12(\psi_y - 2U_2e^{-\psi}) & \frac12(\psi_x - 2U_1e^{-\psi}) \end{pmatrix} \begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix},$$

$$\begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix}_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac12(\psi_y - 2U_2e^{-\psi}) & \frac12(\psi_x - 2U_1e^{-\psi}) \\ e^\psi & -\frac12(\psi_x + 2U_1e^{-\psi}) & \frac12(\psi_y + 2U_2e^{-\psi}) \end{pmatrix} \begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix}.$$

4.2. **A model solution.** If $\mathbf{U} = U\,dz^3 = z^{-(1+k)}\,dz^3$, then $\psi = \log\left(\frac{|\log|z|^2|}{|z|^k}\right)$ satisfies (14).[3] Pass to the upper half plane by setting $z = e^{iw}$ and then

$$\mathbf{U} = \widetilde{U}dw^3 = -ie^{i(2-k)w}dw^3.$$

Now put

$$\alpha \equiv \frac{2-k}{2}.$$

Writing $\widetilde{U} = \widetilde{U}_1 + i\widetilde{U}_2$, we have $\widetilde{U}_1 = \sin(2\alpha x)e^{-2\alpha y}$ and $\widetilde{U}_2 = -\cos(2\alpha x)e^{-2\alpha y}$. The metric is

$$e^\psi|dz|^2 = |\log|z|^2|\,|dz|^2 = e^{\tilde\psi}|dw|^2 = 2ye^{(k-2)y}|dw|^2, \tag{17}$$

where $w = x + iy$. Therefore, with respect to the w coordinate, the conformal factor $e^{\tilde\psi}$ satisfies

$$\tilde\psi = \log(2y) + (k-2)y.$$

In terms of the real variables x, y, the developing map (12), (13) becomes

$$\begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 \\ 2ye^{-2\alpha y} & \frac{1}{2y}\sin 2\alpha x & \alpha - \frac{1}{2y} + \frac{1}{2y}\cos 2\alpha x \\ 0 & -\alpha + \frac{1}{2y} + \frac{1}{2y}\cos 2\alpha x & -\frac{1}{2y}\sin 2\alpha x \end{pmatrix} \begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix},$$

$$\begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix}_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha + \frac{1}{2y} + \frac{1}{2y}\cos 2\alpha x & -\frac{1}{2y}\sin 2\alpha x \\ 2ye^{-2\alpha y} & -\frac{1}{2y}\sin 2\alpha x & -\alpha + \frac{1}{2y} - \frac{1}{2y}\cos 2\alpha x \end{pmatrix} \begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix}.$$

Example 6. *When $k = 2$ the equations become algebraic: $f_{xx} = 2y\xi$, $f_{xy} = \frac1y f_x$, $f_{yy} = 2y\xi$. The solution is*

$$f = E + (Fx + G)y + \left(x^2y + \tfrac13 y^3\right)\xi,$$

where E, F, and G are constant vectors. The condition (5) is satisfied if $\det(F, G, \xi) = 2$.

[3] The transformation $\varphi = \log|U| - \frac12\psi$ results in the condition that $e^{2\varphi}|dz|^2$ has constant curvature -4. We thank R. Bryant for pointing this out.

Let

$$G=(-\cos 2\alpha x+1)f_x+(\sin 2\alpha x)f_y, \qquad H=(-\cos 2\alpha x-1)f_x+(\sin 2\alpha x)f_y.$$

Then we have

$$G_y+\alpha G=2ye^{-2\alpha y}(\sin 2\alpha x)\xi, \qquad H_y=(\tfrac{1}{y}-\alpha)H+2ye^{-2\alpha y}(\sin 2\alpha x)\xi.$$

These equations can be solved to find

$$\begin{aligned} G &= -\tfrac{2}{\alpha}(\sin 2\alpha x)\xi(y+\tfrac{1}{\alpha})e^{-2\alpha y}+K(x)e^{-\alpha y},\\ H &= -\tfrac{2}{\alpha}ye^{-2\alpha y}(\sin 2\alpha x)\xi+L(x)ye^{-\alpha y}. \end{aligned}$$

Here $K(x)$ and $L(x)$ are constants of integration in y. Solve for f_x and f_y to find

$$\begin{aligned} f_x &= -\tfrac{1}{2}ye^{-\alpha y}L(x)-\tfrac{1}{\alpha^2}(\sin 2\alpha x)e^{-2\alpha y}\xi+\tfrac{1}{2}K(x)e^{-\alpha y},\\ f_y &= \tfrac{1}{2}(\cot\alpha x)e^{-\alpha y}K(x)+\tfrac{1}{2}(\tan\alpha x)ye^{-\alpha y}L(x)-\tfrac{1}{\alpha}\xi e^{-2\alpha y}(\tfrac{1}{\alpha}\cos 2\alpha x+\tfrac{1}{\alpha}+2y). \end{aligned}$$

Integrate in y to find

$$\begin{aligned} f &= -\tfrac{1}{2\alpha}(\cot\alpha x)e^{-\alpha y}K(x)-\tfrac{1}{2\alpha}(\tan\alpha x)(y+\tfrac{1}{\alpha})e^{-\alpha y}L(x)\\ &\quad +\tfrac{1}{\alpha^2}\xi e^{-2\alpha y}(\tfrac{1}{2\alpha}\cos 2\alpha x+\tfrac{1}{\alpha}+y)+C(x). \end{aligned}$$

Differentiate in x to find

$$\begin{aligned} f_x &= -K'(x)\tfrac{1}{2\alpha}(\cot\alpha x)(-e^{-\alpha y})+K(x)\tfrac{1}{2}(\csc^2\alpha x)e^{-\alpha y}\\ &\quad -L'(x)\tfrac{1}{2\alpha}(\tan\alpha x)(y+\tfrac{1}{\alpha})e^{-\alpha y}-L(x)\tfrac{1}{2}(\sec^2\alpha x)(y+\tfrac{1}{\alpha}e^{-\alpha y}\\ &\quad -\tfrac{1}{\alpha^2}\xi(\sin 2\alpha x)e^{-2\alpha y}+C'(x). \end{aligned}$$

This can be compared to the formula above for f_x, and we can equate like terms in y:

$$\begin{aligned} ye^{-\alpha y}\text{ terms} &: -\tfrac{1}{2}L(x)=-L'(x)\tfrac{1}{2\alpha}\tan\alpha x-L(x)\tfrac{1}{2}\sec^2\alpha x,\\ e^{-\alpha y}\text{ terms} &: \tfrac{1}{2}K(x)=-K'(x)\tfrac{1}{2\alpha}\cot\alpha x+K(x)\tfrac{1}{2}\csc^2\alpha x\\ &\quad -L'(x)\tfrac{1}{2\alpha^2}\tan\alpha x-L(x)\tfrac{1}{2\alpha}\sec^2\alpha x,\\ e^{-2\alpha y}\text{ terms} &: -\tfrac{1}{\alpha^2}\xi\sin 2x=-\tfrac{1}{\alpha^2}\xi\sin 2x,\\ \text{constant terms} &: C'(x)=0 \end{aligned}$$

So $C(x)=C$ is a constant, and we solve for first $L(x)$ and then $K(x)$ to find

$$L(x)=A\cos\alpha x, \qquad K(x)=(-Ax+B)\sin\alpha x,$$

for constants of integration A and B. Plugging these back, we find

$$\begin{aligned} f &= \tfrac{1}{2\alpha}Ae^{-\alpha y}(x\cos\alpha x - \tfrac{1}{\alpha}\sin\alpha x - y\sin\alpha x) - \tfrac{1}{2\alpha}Be^{-\alpha y}\cos\alpha x \\ &\quad + \tfrac{1}{\alpha^2}\xi e^{-2\alpha y}(\tfrac{1}{2\alpha}\cos 2\alpha x + \tfrac{1}{\alpha} + y) + C, \\ f_x &= -\tfrac{1}{2}Ae^{-\alpha y}(x\sin\alpha x + y\cos\alpha x) + \tfrac{1}{2}Be^{-\alpha y}\sin\alpha x - \tfrac{1}{\alpha^2}\xi e^{-2\alpha y}\sin 2\alpha x, \\ f_y &= -\tfrac{1}{2}Ae^{-\alpha y}(x\cos\alpha x - y\sin\alpha x) + \tfrac{1}{2}Be^{-\alpha y}\cos\alpha x \\ &\quad - \tfrac{1}{\alpha^2}\xi e^{-2\alpha y}(\cos 2\alpha x + 1 + 2\alpha y). \end{aligned}$$

It is straightforward to check that normalization condition (5) is satisfied if

$$\det(A, B, \xi) = -8. \tag{18}$$

5. Monodromy

We now connect our explicit developing map for a parabolic affine sphere to the holomorphic function representation, then to affine coordinates to determine monodromy. The parabolic affine sphere is also a Monge-Ampère manifold. We write the developing map as the graph of Φ with affine normal $\xi = (0, 0, 1)$ In other words, the vector function f satisfies

$$f = (u_1, u_2, \Phi). \tag{19}$$

It is a simple matter to check that the structure equations (7) are solved. Now using the expression of the u_i in terms of holomorphic data and imposing equation (16) above, we find that the cubic form U satisfies

$$U = -2i\det(f_z, f_{zz}, \xi) = \tfrac{1}{2}i\tau_z.$$

Moreover, if $\tilde{U}$ is the cubic form for the dual parabolic affine sphere given by

$$\tilde{f} = (v_2, v_1, \Psi),$$

then we find

$$\tilde{U} = -\tfrac{1}{2}i\tau_z = -U.$$

Remark. In fact, $\widetilde{U} = -U$ is true in all dimensions and reflects a more general phenomenon on all affine Kähler metrics. The Legendre transform gives new affine flat coordinates and a new potential function. The resulting metric is isometric to the original one. The cubic form C in general is the difference between the affine flat connection and the Levi-Civita connection of the metric. It is well known (see e.g. [2]) that the Legendre transform takes C to $-C$. As our U is simply the $(3, 0)$ part of the cubic form C with the up index lowered by the metric, $\tilde{U} = -U$ must be true by these general facts.

If the metric is given by $e^\psi|dz|^2 = \tau_2|dz|^2$ as above, then it is easy to check that $\psi = \log\tau_2$ satisfies equation (14) above.

Example 7. *In our example above, when $k = 0$ we have $U = \frac{1}{z}dz^3$, which naturally leads us to choose $\tau = -2i\log z$. Note that $\tau_2 = -2\log|z|$ is positive near $z = 0$, and so $\tau_2|dz|^2$ is a suitable metric. Moreover, this is the same metric as in equation (17) above, and so this choice of coordinate z, cubic form $U = \frac{1}{z}dz^3$ and metric $-2\log|z|\,|dz|^2$ corresponds to the metric and developing map constructed earlier (for $\alpha = 0$).*

Recall that

$$z = e^{iw} = e^{-y+ix}$$

in terms of the coordinates above. Compute then

$$\begin{aligned}
\phi &= -2iz\log z + 2iz,\\
\chi &= -iz^2\log z + \tfrac{3}{2}iz^2,\\
u_1 &= z_1 = e^{-y}\cos x,\\
u_2 &= -\phi_1 = -\mathrm{Re}(-2iz\log z + 2iz)\\
&= 2e^{-y}(-y\sin x + x\cos x - \sin x),\\
\Phi &= -z_2\phi_1 + \chi_2\\
&= e^{-2y}(1 + \tfrac{1}{2}\cos 2x + y).
\end{aligned}$$

This is then just the parametization of $f = (u_1, u_2, \Phi)$ above in equation (19) for the choices of vectors $A = -4\mathbf{j}$, $B = -2\mathbf{i}$, $\xi = \mathbf{k}$, $C = \mathbf{0}$. Note that these vectors satisfy (18).

Example 8. *When $k = 2$ we have the algebraic solution of Ex. 6. Putting $E = \mathbf{0}$, $F = 2\mathbf{i}$, $G = \mathbf{j}$, and $\xi = \mathbf{k}$, we have $f = (2xy, y, x^2y + \frac{1}{3}y^3) = (v_1, v_2, \Psi)$. This yields $\Psi(v_1, v_2) =$, with $\det(\mathrm{Hess}\,\Psi) = 1$. We note $v_1 = 2xy = \mathrm{Im}(w^2)$, $v_2 = y = \mathrm{Im}(w)$. Indeed $e^\psi = \frac{|\log|z|^2|}{|z|^2}|dz|^2 = 2y|dw|^2 = \tau_2|dw|^2$, where $\tau(w) = 2w$. Then $v_1 = \mathrm{Im}(\phi)$ and $v_2 = \mathrm{Im}(w)$ as expected. Note under $x \to x + 2\pi$, $v_1 \to v_2 + 2\pi v_2$, $v_2 \to v_2$.*

We now calculate the monodromy of the $k = 0$ example about the singularity. The coordinates u_1, u_2 are affine flat with respect to the connection ∇. The affine monodromy of ∇ is may be determined by seeing how u_1 and u_2 change along a loop around the singularity $z = 0$. By inspection, we can see that u_2 is multiple-valued around the singularity (i.e. as x goes from 0 to 2π). For a fixed value of y, we have in the x, y coordinates.

$$\begin{aligned}
f(0,y) &= (0, e^{-y}, e^{-2y}(\tfrac{3}{2}+y)),\\
f(2\pi,y) &= (4\pi e^{-y}, e^{-y}, e^{-2y}(\tfrac{3}{2}+y)),\\
f_x(0,y) &= (-2ye^{-y}, 0, 0),\\
f_x(2\pi,y) &= (-2ye^{-y}, 0, 0),\\
f_y(0,y) &= (0, -e^{-y}, -2e^{-2y}(1+y)),\\
f_y(2\pi,y) &= (-4\pi e^{-y}, -e^{-y}, -2e^{-2y}(1+y)).
\end{aligned}$$

The monodromy of the connection ∇ around a loop around the singularity at $z=0$ is computed by integrating the initial value problem to find $f=(u_1,u_2,\Phi)$. Let $\mathrm{dev}=(u_1,u_2)$ (this is the developing map of the affine flat structure—see e.g. [20]). In the w coordinates, this means to calculate how dev changes as $x \mapsto x+2\pi$. In general, there is a coordinate change

$$\mathrm{dev} \mapsto M\,\mathrm{dev} + N$$

for $M \in \mathbf{GL}(2,\mathbb{R})$, $N \in \mathbb{R}^2$. Differentiate this coordinate change in terms of x and y, and then plug in at some point $(0,y)$, $y \gg 0$, to find the equations

$$\begin{aligned}
\mathrm{dev}(2\pi,y) &= M\,\mathrm{dev}(0,y) + N,\\
\mathrm{dev}_x(2\pi,y) &= M\,\mathrm{dev}_x(0,y),\\
\mathrm{dev}_y(2\pi,y) &= M\,\mathrm{dev}_y(0,y).
\end{aligned}$$

The computations above imply that

$$M = \begin{pmatrix} 1 & \frac{2\pi}{y} \\ 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This action is simpler in the coordinates u_i. If we define $\partial_i \equiv \frac{\partial}{\partial u_i}$ then under $x \mapsto x+2\pi$, $y \mapsto y$, we find $\partial_1 \mapsto \partial_1 - 4\pi\partial_2$, $\partial_2 \mapsto \partial_2$, the linear part of the monodromy given by an $SL(2,\mathbb{Z})$ matrix. The translation part (represented by N above) is 0.

6. A Radial Model for Parabolic Affine Spheres

In this section we generalize the model of Baues and Cortés for parabolic affine spheres to two other cases. Recall the standard parabolic affine sphere has potential $\Phi = \|x\|^2/2$ and can be thought of in terms of a cone over the standard elliptic affine sphere $\|x\|=1$. Baues and Cortés show that this can be done with any elliptic affine sphere.

The first generalization still involves a cone over an elliptic affine sphere. For the standard elliptic affine sphere, we recover the radially symmetric model in $\mathbb{R}^{n+1}$ (3)

$$\Phi = \int (r^{n+1} + 1)^{\frac{1}{n+1}} dr.$$

We present an analogue of this for any elliptic affine sphere.

The second generalization is for part of a cone (for radial coordinate r close to 0) over a hyperbolic affine sphere of even dimension. (We want the hyperbolic affine sphere to have dimension 2, so the ambient space has dimension 3.)

The motivation for this section is to find more local models of parabolic affine spheres in dimension 3. It may take considerable flexibility of the local models in order to piece together appropriate global models of parabolic affine spheres on an affine structure on S^3 minus singularities.

The model for these calculations may be found in [18].

Consider an affine Kähler metric

$$g_{ij}\, dx^i dx^j = \frac{\partial^2 \Phi}{\partial x^i \partial x^j}\, dx^i dx^j$$

on a domain in $\mathbb{R}^{n+1}$. Let

$$X = x^i \frac{\partial}{\partial x^i}$$

be the radial vector field. Assume the metric potential Φ satisfies the ansatz

$$x^i\, \frac{\partial \Phi}{\partial x^i} = X\Phi = f(\Phi). \tag{20}$$

This will ensure that under g_{ij}, the hypersurfaces $\{\Phi = c\}$ are orthogonal to the radial direction X.

Apply $\frac{\partial}{\partial x^j}$ to (20) to get

$$x^i\, \frac{\partial^2 \Phi}{\partial x^i \partial x^j} = \left[\frac{df}{d\Phi} - 1\right] \frac{\partial \Phi}{\partial x^j} \quad \text{for} \quad j = 1, \dots, n. \tag{21}$$

Consider a vector Y tangent to the hypersurface $H_c = \{\Phi = c\}$. (So $Y\Phi = 0$.) Then (21) shows that

$$g(X, Y) = x^i \frac{\partial^2 \Phi}{\partial x^i \partial x^j} y^j = \left[\frac{df}{d\Phi} - 1\right] \frac{\partial \Phi}{\partial x^j} y^j = \left[\frac{df}{d\Phi} - 1\right] Y\Phi = 0,$$

and also

$$g(X, X) = x^i \frac{\partial^2 \Phi}{\partial x^i \partial x^j} x^j = \left[\frac{df}{d\Phi} - 1\right] X\Phi = f\left[\frac{df}{d\Phi} - 1\right].$$

Now we'll show that the g restricts to a constant multiple of the centroaffine metric on H_c.

Let D be the canonical flat connection on $\mathbb{R}^{n+1}$. Then our affine Kähler metric g is given by

$$g(A,B) = (D_A d\Phi, B) \tag{22}$$

where A, B are vectors and $(\cdot\,,\cdot)$ is the pairing between one forms and vectors. Assume X is transverse to H_c. So at $x \in H_c$, $\mathbb{R}^{n+1} = T_x(\mathbb{R}^{n+1})$ splits into $T_x(H_c) \oplus \langle X \rangle$. Then we have

$$D_Y Z = \nabla_Y Z + h(Y,Z)X \tag{23}$$

where Y, Z are tangent vectors to H_c, ∇ is a connection on $T(H_c)$, and the centroaffine second fundamental form h is a symmetric $(0,2)$ tensor on H_c.

Now consider

$$\begin{aligned} 0 &= Y(d\Phi, Z) \\ &= (d\Phi, D_Y Z) + (D_Y d\Phi, Z) \\ &= f\,h(Y,Z) + g(Y,Z) \end{aligned}$$

by (20), (22) and (23). Therefore, $g(Y,Z) = -f\,h(Y,Z)$ for Y, Z tangent to H_c. All together we have

Proposition 1. *Under the metric g, each level set of the potential Φ is perpendicular to the radial direction X. Furthermore,*

$$g(X,X) = f\left[\frac{df}{d\Phi} - 1\right], \qquad g(Y,Z) = -f\,h(Y,Z),$$

where Y and Z are tangent to H_c and h is the centroaffine second fundamental form of H_c.

It is useful to introduce a radial parameter r. Let $H = H_1$ and foliate a region in $\mathbb{R}^{n+1}$ by

$$\bigcup_{r>0} rH.$$

Our ansatz (20) implies that the potential $\Phi = \Phi(r)$ and

$$\frac{d\Phi}{dr} = \frac{1}{r} X\Phi = \frac{f}{r}, \tag{24}$$

and that each rH is a level set of Φ.

Choose a basis of tangent vectors $\{Y_i\}_{i=1}^n$ to H so that

$$\det(X, Y_1, \dots, Y_n) = 1. \tag{25}$$

We are interested in the case g is a parabolic affine sphere metric. In other words, we require for a positive constant k,

$$\begin{aligned} k &= \det_{1\le i,j\le n+1} g_{ij} \\ &= g(X,X) \det_{1\le i,j\le n} g(Y_i,Y_j) \\ &= f\left[\frac{df}{d\Phi}-1\right](-f)^n \det_{1\le i,j\le n} h(Y_i,Y_j) \end{aligned} \tag{26}$$

This is because (25) and Proposition 1.

We will solve this differential equation for f as a function of r. To achieve this, we need to see how $\det h(Y_i,Y_j)$ scales. Let X denote the position vector for a point in $H=H_1$. Then $\tilde X = rX$ is on another level set H_c, and if $\tilde Y_i = r^{-\frac{1}{n}} Y_i \circ \mathcal{S}$, then $\tilde X, \tilde Y_i$ satisfy (25). (The scaling transformation $\mathcal{S}\colon X \mapsto \frac{1}{r}X$.) Then plug into (23)

$$\begin{aligned} D_{\tilde Y_i}\tilde Y_j &= (\tilde Y_i, d\tilde Y_j) \\ &= r^{-\frac{2}{n}}(Y_i, \mathcal{S}^* dY_j) \\ &= r^{-\frac{n+2}{n}} D_{Y_i} Y_j \\ &= r^{-\frac{n+2}{n}}[\nabla_{Y_i}Y_j + h(Y_i,Y_j)X] \\ &= r^{-\frac{n+2}{n}}\nabla_{Y_i}Y_j + r^{-\frac{2n+2}{n}} h(Y_i,Y_j)\tilde X \end{aligned}$$

Therefore, $h(\tilde Y_i,\tilde Y_j) = r^{-\frac{2n+2}{n}} h(Y_i,Y_j)$ and

$$\det_{1\le i,j\le n} h(\tilde Y_i,\tilde Y_j) = r^{-2n-2} \det_{1\le i,j\le n} h(Y_i,Y_j).$$

Our differential equation is then

$$\begin{aligned} Kr^{2n+2} &= f^{n+1}\left[\frac{df}{d\Phi}-1\right] \\ &= f^{n+1}\left[\frac{df}{dr}\Big/\frac{d\Phi}{dr}-1\right] \\ &= rf^n\frac{df}{dr} - f^{n+1} \end{aligned}$$

by (24). On $H=H_1$, K is the constant $(-1)^n k/\det h(Y_i,Y_j)$. K is constant on H by (26), and we extend K to be constant on the whole cone $\bigcup_{r>0} rH$. We have taken care of the scaling by using the factor r^{2n+2}.

If $f = r^2\psi$ then the equation is

$$\psi^{n+1} + r\psi^n\frac{d\psi}{dr} = K,$$

which can be solved so that for a real constant A,

$$\begin{aligned}
\psi &= \left(K + Ar^{-n-1}\right)^{\frac{1}{n+1}}, \\
f &= r^2\left(K + Ar^{-n-1}\right)^{\frac{1}{n+1}}, \\
\Phi &= \int r\left(K + Ar^{-n-1}\right)^{\frac{1}{n+1}} dr \\
&= \int \left(Kr^{n+1} + A\right)^{\frac{1}{n+1}} dr.
\end{aligned} \tag{27}$$

So far we've just dealt with the radial direction. Now we see what conditions this ansatz imposes on the hypersurfaces H_c. We are interested near the origin (as $r \to 0^+$).

Since $g(Y, Z) = -f\,h(Y, Z)$, then we must require the centroaffine second fundamental form h to be definite in order for g to be positive definite. This means that the hypersurfaces H_c must be locally strictly convex. If H_c points away from the origin (i.e. if the origin and the hypersurface are on opposite sides of the tangent plane), then h is positive definite. If H_c points toward the origin (i.e. if the origin and the hypersurface are on the same side of the tangent plane), then h is negative definite.

We apply the technique in Nomizu-Sasaki [22, p. 45] to show that X is a multiple of the affine normal and therefore H_c is an affine sphere. The technique gives a formula for constructing the affine normal ξ to a hypersurface H_c given a transverse vector field X. We will find a function ϕ and a tangent vector field W so that $\xi = \phi X + W$. The technique is this: First compute

$$\begin{aligned}
\phi &= \left| \det_{1\le i,j\le n} (h(Y_i, Y_j)) \right|^{\frac{1}{n+2}} \\
&= \left| k f^{-n-1} \left[\frac{df}{d\Phi} - 1\right]^{-1} \right|^{\frac{1}{n+2}}
\end{aligned}$$

for Y_i chosen as above in (25). Then—since the transverse vector field X is equiaffine—we have

$$W = w^i \frac{\partial}{\partial t^i} = -h^{ij}\frac{\partial \phi}{\partial t^j}\frac{\partial}{\partial t^i} = 0. \tag{28}$$

Here h^{ij} is the inverse matrix of the second fundamental form $h_{ij} = h\left(\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}\right)$ as in (23). The formula follows since ϕ is constant on H_c. Therefore, the affine normal $\xi = \phi X$ and H_c is a proper affine sphere. Assuming it's locally strictly convex, it is an elliptic affine sphere if it points toward the origin and a hyperbolic affine sphere if it points away from the origin.

Now consider different cases for signs of A in our exact solution (27) above. To evaluate the metric we will need below

$$\begin{aligned} g(X,X) &= f\left[\frac{df}{d\Phi}-1\right] \\ &= f\left[\frac{df}{dr}\Big/\frac{d\Phi}{dr}-1\right] \\ &= f\left[\frac{2K+Ar^{-n-1}}{K+Ar^{-n-1}}-1\right] \\ &= r^2K(K+Ar^{-n-1})^{-\frac{n}{n+1}}. \end{aligned}$$

Case 1: $\mathbf{A}>\mathbf{0}$. In this case as $r\to 0^+$, $f>0$, $g(X,X)>0$ as long as $K>0$, and

$$g(Y,Z) = -f\,h(Y,Z).$$

Therefore, we must have h negative definite (which makes $K>0$). In this case, the hypersurface H points toward the origin and we have an elliptic affine sphere.

Example 9. *For the standard elliptic affine sphere $H=\{\|x\|=r=1\}$, this example is the radially symmetric example (3).*

Case 2: $\mathbf{A}=\mathbf{0}$. In this case

$$g(Y,Z) = -r^2K^{\frac{1}{n+1}}\,h(Y,Z), \qquad g(X,X)=r^2K^{\frac{1}{n+1}}.$$

So we must have $K=(-1)^n k/\det h(Y_i,Y_j)>0$ and h negative definite (which implies $K>0$). So then the H_c are elliptic affine spheres. This is the case that Baues and Cortés did.

Example 10. *The standard parabolic affine sphere comes from the potential $\Phi=\frac{1}{2}\|x\|^2=\frac{1}{2}\sum(x^i)^2$. In this case we have*

$$\begin{aligned} g_{ij} &= \delta_{ij}, \\ H &= \{\|x\|^2=2\}, \\ \Phi &= \frac{1}{2}r^2, \\ f &= 2\Phi = r^2, \\ k &= 1, \\ K &= 1, \\ g(X,X) &= 2\Phi = r^2. \end{aligned}$$

We may easily check that this example satisfies all the equations above.

To be concrete, a function $u : \mathbb{R}^n \to \mathbb{R}$ *obeying* $\det(\mathrm{Hess}(u)) = u^{-(n+2)}$ *determines an elliptic affine sphere,* S*, given by* $x \mapsto y(x) = (x, u(x))$*. Then the mapping* $(r, x) \mapsto y(r, x) = (rx, ru(x))$ *is locally invertible and thus we can write* $r = r(y)$*. Then* $\Phi(y) = r(y)^2/2$ *is a Hessian potential for a parabolic affine sphere (i.e.* $\det(\mathrm{Hess}(\Phi)) = 1$*) with cone metric over* S*. In the* (r, x) *coordinates we have* $ds^2 = dr^2 + r^2 u_{ij} dx^i dx^j$*, a cone over* S*.*

Case 3: A < 0. In this case as $r \to 0^+$, we must have n even in order for (27) to make sense. (We are interested in the case $n = 2$.) Then we must have

$$K = (-1)^n k / \det h(Y_i, Y_j) > 0$$

to make $g(X, X) > 0$. For n even, this happens when h is either positive or negative definite. Also, $f < 0$ for $r \to 0^+$. Therefore, $g(Y, Z) = -f\, h(Y, Z)$ implies that h must be positive definite. In other words, H points away from the origin and is a hyperbolic affine sphere.

Example 11. *For this case, consider* $n = 2$ *and the standard hyperbolic affine sphere in* $\mathbb{R}^3$

$$H = \{x^3 = \sqrt{1 + (x^1)^2 + (x^2)^2}\}.$$

Then $r = \sqrt{(x^3)^2 - (x^1)^2 - (x^2)^2}$*. Let* $A = -1$ *and* $K = 1$*. Then*

$$\Phi = \int \left(r^3 - 1\right)^{\frac{1}{3}} dr.$$

Compute the metric $g_{ij} = \dfrac{\partial^2 \Phi}{\partial x^i \partial x^j}$ *to be*

$$\frac{1}{r^3(r^3-1)^{\frac{2}{3}}} \begin{pmatrix} -r^5 + r^2 + (x^1)^2 & x^1 x^2 & -x^1 x^3 \\ x^1 x^2 & -r^5 + r^2 + (x^2)^2 & -x^2 x^3 \\ -x^1 x^3 & -x^2 x^3 & r^5 - r^2 + (x^3)^2 \end{pmatrix}.$$

It is straightforward to check that $\det g_{ij} = 1$*. Also,*

$$g(X, X) = r^2(1 - r^{-3})^{-\frac{2}{3}}, \qquad g(Y, Z) = -r^2(1 - r^{-3})^{\frac{1}{3}}\, h(Y, Z).$$

So this metric is positive definite only for $0 < r < 1$*. It is incomplete as* $r \to 0$ *and as* $r \to 1$*.*

7. Elliptic Affine Spheres and the "Y" Vertex

In this section we will prove the existence of elliptic affine two-sphere metrics with singularities – first locally near a singularity, then globally on S^2 minus three points. The metric cone yields a parabolic affine sphere metric near the "Y" vertex.

7.1. **Local Analysis.** It is a simple matter to mimic the construction of section 4 in the case of an elliptic affine sphere, for which the normal vector can be taken to point to the origin: $\xi = -f$. The second structure equation in (7) is then $D_X\xi = -f_*X$. With this modification, the integrability condition (Eq. (14)) becomes $U_{\bar{z}} = 0$ and

$$\psi_{z\bar{z}} + |U|^2 e^{-2\psi} + \frac{1}{2}e^{\psi} = 0. \tag{29}$$

Since we can construct a parabolic affine sphere on the cone over an elliptic sphere, a solution to this equation on the thrice-punctured sphere will lead to a parabolic affine sphere on $\mathbb{R}^3$ minus a Y-shaped set – whence a semiflat special Lagrangian torus fibration over this base. We will prove that a solution ψ exists to Eq. (29) in a certain function space. This will allow us to study monodromy, as in section 5. We then discuss the more global setting of the thrice-punctured sphere.

For definiteness, we consider the case $U = z^{-2}$ and make the ansatz $\psi = \psi(|z|)$. We look near $z = 0$, so we make the change of variables $t = -\log|z|$, $t \in (T, \infty)$, $T \gg 0$. This leads to the equation

$$N(\psi) := \partial_t^2\psi + 4e^{-2(\psi-t)} + 2e^{\psi-2t} = 0. \tag{30}$$

We put $\psi = \psi_0 + \phi$, where $\psi_0 = t + \log(2t)$ is the solution to the parabolic equation (14). Note that the last term is $O(te^{-t})$ for this function. We want to solve $N(\psi_0 + \phi) = 0$, which we expand as

$$N(\psi_0 + \phi) = N(\psi_0) + dN(\phi)|_{\psi_0} + Q(\phi)|_{\psi_0}, \tag{31}$$

where $Q(\phi)$ contains quadratic and higher terms. Explicitly,

$$Q(\phi) = \frac{1}{t^2}\left(e^{-2\phi} - (1 - 2\phi)\right) + 4te^{-t}\left(e^{\phi} - (1 + \phi)\right).$$

Note that Q is not even a differential operator. One calculates

$$N(\psi_0) = 4te^{-t},$$

$$dN(\phi)|_{\psi_0} =: L\phi := \left[\partial_t^2 + V(\psi_0)\right]\phi,$$

where

$$V(\psi_0) = -8e^{-2(\psi_0-t)} + 2e^{\psi_0-2t} = -\frac{2}{t^2} + 4te^{-t}.$$

Thus $L\phi = (\partial_t^2 - \frac{2}{t^2} + 4te^{-t})\phi = (L_0 + 4te^{-t})\phi$, where $L_0 = \partial_t^2 - \frac{2}{t^2}$. The equation (29) is now

$$L\phi = f - Q(\phi),$$

with $f = -4te^{-t}$. The idea will be to find an appropriate Green function G for L, in terms of which a solution to this equation becomes a fixed point of the mapping $\phi \to G(f - Q(\phi))$—then to find a range of ϕ

where this is a contraction map, whence a solution by the fixed point theorem.

We claim that this map is a contraction for $\phi \sim O(te^{-t})$. More specifically, consider for a value of $T > 2$ to be determined later, the Banach space $\mathcal{B}$ of continuous functions on $[T, \infty)$ with norm

$$\|f\|_{\mathcal{B}} = \sup_{t \geq T} \frac{f(t)}{te^{-t}}.$$

Showing the map $\phi \to G(f - Q(\phi))$ is a contraction map involves estimating Gf and $GQ\phi$. In fact, since Q is a quadratic, nonderivative operator, it is easy to see that $Q\phi$ is order te^{-t} (even smaller). We then show that G preserves the condition $O(te^{-t})$ by showing $Gf \in \mathcal{B}$ (recall that $f \in \mathcal{B}$, too). To find G, we write $L = L_0 - \delta_L$, where $\delta_L = -4te^{-t}$, so that $G = L^{-1} = L_0^{-1} + L_0^{-1}\delta_L L_0^{-1} + ...$ To solve the equation $Lu = f$, we first note that the change of variables $v = u + 1$ leads to the equation $Lv = -\frac{2}{t^2}$. Let $v_0 = L_0^{-1}(-\frac{2}{t^2}) = 1$. Then define $v_{k+1} = L_0^{-1}\delta_L v_k$. Then $v = \sum_{k=0}^{\infty} v_k$ and $Gf = u = \sum_{k=1}^{\infty} v_k$.

Lemma 2. $|v_k(t)| < (16te^{-t})^k$ *pointwise.*

Proof. It is true for $k = 0$. To compute v_{k+1} one solves the differential equation by the method of variation of parameters,[4] using the homogeneous solution t^2 or t^{-1}. We have

$$v_{k+1}(t) = t^2 \int_t^{\infty} t_1^{-4} \int_{t_1}^{\infty} t_2^2(-4t_2 e^{-t_2} v_k(t_2) dt_2 dt_1.$$

One computes $v_1(t) = -4(t+2+2/t)e^{-t}$, and therefore $|v_1(t)| < 16te^{-t}$ for $t > 2$. Now assume $|v_k(t)| < (16te^{-t})^k$ for some $k \geq 1$. First compute for $a, b \in \mathbb{N}$, $t > a/b$ and $t > 2$:

$$\begin{aligned}\int_t^{\infty} s^a e^{-bs} ds &= -\frac{1}{b}\left[s^a + \frac{a}{b}s^{a-1} + \frac{a(a-1)}{b^2}s^{a-2} + ...\right] e^{-bs}\Big|_t^{\infty} \\ &\leq \frac{1}{b}t^a(1 + (a/bt) + (a/bt)^2 + ...)e^{-bt} \\ &\leq \frac{t}{b(t-1)}t^a e^{-bt} \leq 2t^a e^{-bt}.\end{aligned}$$

[4]We can write $G_0 h(t) = \int_T^{\infty} K_0(t,s)h(s)ds$, where $K_0(t,s) = \frac{1}{3}(\frac{s^2}{t} - \frac{t^2}{s})$ for $s > t$ and zero otherwise (this form of the kernel is relevant to the condition of good functional behavior at infinity). One can also use the equivalent $G_0 h(t) = t^2 \int_t^{\infty} t_1^{-4} \int_{t_1}^{\infty} s^2 h(t_2) dt_2 dt_1$, which appears in the text.

Therefore,

$$
\begin{aligned}
|v_{k+1}(t)| &\leq 4t^2 \int_t^\infty t_1^{-4} \int_{t_1}^\infty t_2^3 e^{-t_2} v_k(t_2) dt_2 dt_1 \\
&\leq 4t^2 \int_t^\infty t_1^{-4} \int_{t_1}^\infty t_2^3 e^{-t_2} (16)^k t_2^k e^{-kt_2} dt_2 dt_1 \\
&\leq (16)^k \cdot 4 \cdot 2t^2 \int_t^\infty t_1^{k+3-4} e^{-(k+1)t_1} dt_1 \\
&\leq (16)^{k+1} t^{k+1} e^{-(k+1)t}
\end{aligned}
$$

and the lemma is proven.[5] □

It now follows that $u < C(1-4te^{-t})^{-1}4te^{-t} \leq C'te^{-t}$ for some constant C'. The space $S = \{f(t) : |f(t)| \leq 2C'te^{-t}\}$ forms a closed subset of $\mathcal{B}$ on which we apply the contraction mapping theorem.

Proposition 3. *There is a constant $T > 0$ so that for $t \geq T$, the equation (29) has a solution of the form* $\log(2t) + t + O(te^{-t})$.

Proof. We now show the mapping $\phi \to A\phi \equiv Gf - GQ\phi$ is a contraction. First, Gf lies within S and $Q\phi$ is small since Q is a quadratic nondifferential operator. More specifically, for some $T > 0$, the sup norm of ϕ on $[T, \infty)$ can be made arbitrarily small. Therefore, on $[T, \infty)$, $\|Q\phi\|_{\mathcal{B}} \ll \|\phi\|_{\mathcal{B}}$. As a result, A maps S to S. Further, note $\|A\phi_1 - A\phi_2\|_{\mathcal{B}} = \|GQ\phi_1 - GQ\phi_2\|_{\mathcal{B}} = \|G(Q\phi_1 - Q\phi_2)\|_{\mathcal{B}}$, since G is linear. Since Q is quadratic and ϕ is small, $\|Q\phi_1 - Q\phi_2\|_{\mathcal{B}} \ll \|\phi_1 - \phi_2\|_{\mathcal{B}}$, and A is a contraction since in the operator norm $\|G\| < C'/4$ by the previous lemma. Thus we can clearly find a $T > 0$ so that for a fixed $\theta < 1$, $\|A\phi_1 - A\phi_2\|_{\mathcal{B}} \leq \theta\|\phi_1 - \phi_2\|_{\mathcal{B}}$ for $\phi_1, \phi_2 \in S$. By the fixed point theorem, there exists ϕ such that $A\phi = \phi$. ϕ is smooth by standard bootstrapping. Then $\log(2t) + t + \phi(t)$ solves (29). □

In the next section, we will require the local form of our global function to be consistent with the $\log|\log|z|^2| - \log|z| + O(|z|\log|z|)$ behavior of this local solution. The dominant term is $-\log|z|$ which comes from the form of U and determines the residue at the singularity.

7.2. **Global Existence.** The coordinate-independent version of Eq. (29) for a general background metric, is

$$
\Delta u + 4\|U\|^2 e^{-2u} + 2e^u - 2\kappa_0 = 0 \tag{32}
$$

on S^2, where norms, gradients, integrals, etc., are taken with respect to the background metric. U is a holomorphic cubic differential, which we take to have exactly 3 poles of order 2 and thus no zeroes, and u is

[5]Note that we needed $k - 1 \geq 0$ to bound a/b from above. That the $k = 1$ term is the proper order follows from some fortuitous cancellation.

taken to have a prescribed singularity structure such that $\int \Delta u = 6\pi$, which follows from our local analysis in Section 7.1.

Near each pole of U, there is a local coordinate z so that the pole is at $z = 0$, and $U = z^{-2}dz^3$ exactly. We call this z the *canonical holomorphic coordinate.* In a neighborhood of each pole, we take

$$u_0 = \log|\log|z|^2| - \log|z| \tag{33}$$

and the background metric to be $|dz|^2$. The background metric and u_0 are extended smoothly to the rest of $\mathbb{CP}^1$. Note that $\int \Delta u_0 = 6\pi$ (each pole contributes 2π). All integrals in this section will be evaluated with respect to the background metric.

To implement the required singularity structure, we write $u = u_0 + \eta$ for η in the Sobolev space H_1. Note this implies $\int \Delta\eta = 0$. We define the functional

$$\begin{aligned} J(\eta) \;\; = \;\; & \int \left(\tfrac{1}{2}|\nabla\eta|^2 + (2\kappa_0 - \Delta u_0)\eta + \tfrac{1}{2}3\cdot 4\|U\|^2 e^{-2u_0}e^{-2\eta}\right) \\ & -2\pi\log\int\left(4\|U\|^2e^{-2u_0}e^{-2\eta} + 2e^{u_0}e^{\eta}\right). \end{aligned} \tag{34}$$

(We note that it is necessary to separate η from u_0 as ∇u_0 is not in L^2.) J is not defined for all functions $\eta \in H_1$. One problem is that $\Delta u_0 \notin L^2$. The term $\int \Delta u_o \eta$ can be taken care of by integrating by parts (see the proof of Proposition 5 below). A more serious problem is that $4\|U\|^2 e^{-2u_0} \notin L^p$ for any $p > 1$. This cannot be fixed by integrating by parts, as the example $\eta = -\frac{1}{2}\log|\log|z|^2| \in H_{1,\mathrm{loc}}$ shows. That said, there is a uniform lower bound on J among all $\eta \in H_1$ so that $\int 4\|U\|^2e^{-2u_0}e^{-2\eta} < \infty$ (see the remark after Proposition 5). Thus we can still talk of taking sequences of $\eta \in H_1$ to minimize J. (The term $\int 2e^{u_0}e^{\eta}$ is always finite for $\eta \in H_1$ since $e^{u_0} \in L^p$ for $p < 2$ and Moser-Trudinger shows that $e^{\eta} \in L^q$ for all $q < \infty$.)

We wish to show that $J(\eta)$ has a minimum. If so, then the minimum satisfies the Euler-Lagrange equation

$$\Delta\eta - (2\kappa_0 - \Delta u_0) + 3\cdot 4\|U\|^2e^{-2u_0}e^{-2\eta} + \frac{-2\cdot 4\|U\|^2e^{-2u_0}e^{-2\eta} + 2e^{u_0}e^{\eta}}{\frac{1}{2\pi}\int 4\|U\|^2e^{-2u_0}e^{-2\eta} + 2e^{u_0}e^{\eta}} = 0. \tag{35}$$

One can easily check by integrating this equation that for a solution η_0, the denominator in the last term must be equal to one. Thus $u = \eta_0 + u_0$ satisfies the original equation (32). In this case, the equation η_0 satisfies is

$$\Delta\eta - (2\kappa_0 - \Delta u_0) + 4\|U\|^2e^{-2u_0}e^{-2\eta} + 2e^{u_0}e^{\eta} = 0. \tag{36}$$

This is equivalent to equation (29), the equation for the metric of an elliptic affine sphere: for the background metric h, write $e^{u_0+\eta}h = e^{\psi}|dz|^2$. Then η satisfies (36) if and only if ψ satisfies (29).

Definition 1. *We call η* admissible *if $\eta \in H_1$ and $\int 4\|U\|^2 e^{-2u_0} e^{-2\eta} < \infty$.*

In order to analyze the functional J, for an admissible η, consider $J(\eta+k)$ for k a constant. $J(\eta+k)$ has the form

$$\text{(indep. of } k) + 2\pi k + 3\pi A e^{-2k} - 2\pi \log[2\pi(Ae^{-2k} + Be^{k})],$$

where

$$A = A(\eta) \equiv \frac{1}{2\pi}\int 4\|U\|^2 e^{-2u_0}e^{-2\eta}, \quad B = B(\eta) \equiv \frac{1}{2\pi}\int 2e^{u_0}e^{\eta}. \tag{37}$$

Thus upon setting $(\partial/\partial k)J(\eta+k) = 0$, we find a critical point only if $Ae^{-2k} + Be^{k} = 1$, and this can only happen if

$$AB^2 \le \frac{4}{27}.$$

If $AB^2 > 4/27$, then the infimum occurs as $k \to +\infty$, and if $AB^2 < 4/27$, there are two finite critical points: a local minimum for which $B(\eta+k) = Be^k < 2/3$ and a local maximum for which $B(\eta+k) > 2/3$. With that in mind, we formulate the following variational problem:

Let

$$Q = \{\eta \in H_1 : A + B \le 1\}.$$

We will minimize J for $\eta \in Q$. Note that this will avoid the potential problem at $k \to +\infty$, where $B(\eta+k) \to +\infty$. Also, the inequality in the definition of Q will be important. It will allow us to use the Kuhn-Tucker conditions to control the sign of the Lagrange multiplier in the Euler-Lagrange equations. The discussion above about adding a constant k can be summarized in

Lemma 4. *If $\eta \in Q$, then the minimizer of*

$$\{J(\eta+k) : k \text{ constant}, \eta + k \in Q\}$$

occurs for k so that $A(\eta+k)+B(\eta+k) = 1$, $B(\eta+k) \le 2/3$, and $k \le 0$. If $A(\eta)+B(\eta) < 1$, then the minimizer $k < 0$ and $B(\eta+k) < 2/3$. Moreover, if $A(\eta)+B(\eta) = 1$ and $B(\eta) \le 2/3$, then $k = 0$.

Proof. Compute $(\partial/\partial k)J(\eta+k)$ and use the first derivative test. □

Proposition 5. *There are positive constants γ and R so that for all $\eta \in Q$,*

$$J(\eta) \ge \gamma \int |\nabla \eta|^2 - R.$$

Remark. We can also prove the same result for all admissible $\eta \in H_1$. In this case, we must also control potential minimizers at $k = +\infty$. For admissible $\rho \in H_1$ so that $\int \rho = 0$, consider the functional

$$\tilde{J}(\rho) = \lim_{k\to\infty} J(\rho + k).$$

We bound $\tilde{J}$ from below much the same as the following argument, although there also is an extra term in $\tilde{J}$ that must be handled using the Moser-Trudinger estimate.

Proof. As above, $u_0 = \log|\log|z|^2| - \log|z|$ in the canonical coordinate z near each pole of U. Since $\Delta u_0 \notin L^2$, we should integrate by parts to handle the $-\int \Delta u_0\, \eta$ term in J. Let $u_0' = \log|\log|z|^2|$ near each pole of U and smooth elsewhere. Then $\Delta u_0 = \Delta u_0'$ near each pole and the difference $\Delta u_0 - \Delta u_0'$ is smooth on $\mathbb{CP}^1$. Then if we let ζ be the smooth function $2\kappa_0 - \Delta(u_0 - u_0')$,

$$\begin{aligned} J(\eta) &= \int [\frac{1}{2}|\nabla\eta|^2 + \zeta\eta - \Delta u_0'\, \eta] + 3\pi A - 2\pi \log 2\pi(A+B) \\ &> \int [\frac{1}{2}|\nabla\eta|^2 + \zeta\eta + \nabla u_0' \cdot \nabla\eta] - 2\pi\log 2\pi \\ &\geq C + \int \left[\frac{1}{2}|\nabla\eta|^2 - \frac{1}{4\epsilon}\zeta^2 - \epsilon\eta^2 - \frac{1}{4\epsilon}|\nabla u_0'|^2 - \epsilon|\nabla\eta|^2\right] \\ &\geq C_\epsilon + \int \left(\frac{1}{2} - \delta\right)|\nabla\eta|^2 \end{aligned}$$

Here $\delta = (\frac{1}{\lambda_1} + 1)\epsilon$, for λ_1 the first nonzero eigenvalue of the Laplacian of the background metric, and we've used the facts that $A > 0$ and $A + B \leq 1$. □

Here is another useful lemma.

Lemma 6. *For any $\eta \in H_1$,*

$$AB^2 \geq L = 2\pi^{-3}\left(\int \|U\|^{\frac{2}{3}}\right)^3.$$

If $AB^2 = L$, then there is a constant C such that

$$\eta = C + \tfrac{2}{3}\log\|U\| - u_0.$$

Proof. Let $f = (4\|U\|^2)^{\frac{1}{3}} e^{-\frac{2}{3}(u_0+\eta)}$, $g = e^{\frac{2}{3}(u_0+\eta)}$. Apply Hölder's inequality $\int fg \leq \|f\|_3\|g\|_{\frac{3}{2}}$. The last statement follows from the case of equality in Hölder's inequality. □

Remark. The bound L in the previous lemma does not depend on the background metric; it depends only on the conformal structure on $\mathbb{CP}^1$ and the cubic form U.

An admissible $\eta \in H_1$ is a *weak solution* of (36) if η is a solution of (36) in the sense of distributions.

Proposition 7. *Assume that U is such that $L < 4/27$. Then any minimizer η of $\{J(\eta) : \eta \in Q\}$ is a weak solution of (36).*

Proof. Recall $Q = \{\eta : A + B \le 1\}$.

Case 1: The minimizer η satisfies $A + B < 1$. Since the constraint $A + B \le 1$ is slack, η must satisfy the Euler-Lagrange equation (35). Then as above, we may integrate to find that the denominator $A + B$ in (35) must be equal to 1. Thus this case cannot occur.

Case 2: The minimizer η satisfies $A + B = 1$. In this case, we have Lagrange multipliers $[\mu_0, \mu_1] \in \mathbb{RP}^1$ so that η weakly satisfies

$$\mu_0 \left[\Delta\eta - (2\kappa_0 - \Delta u_0) + 3 \cdot 4\|U\|^2 e^{-2u_0} e^{-2\eta} + \frac{-2 \cdot 4\|U\|^2 e^{-2u_0} e^{-2\eta} + 2e^{u_0} e^{\eta}}{A + B} \right]$$

$$= \mu_1(-2 \cdot 4\|U\|^2 e^{-2u_0} e^{-2\eta} + 2e^{u_0} e^{\eta}),$$

and $A + B = 1$. Thus,

$$\mu_0[\Delta\eta - (2\kappa_0 - \Delta u_0) + a + b] = \mu_1(-2a + b) \tag{38}$$

for

$$a = 4\|U\|^2 e^{-2u_0} e^{-2\eta}, \qquad b = 2e^{u_0} e^{\eta}.$$

Note then that $A = \int a/2\pi$, $B = \int b/2\pi$.

Also note the constraint the Kuhn-Tucker conditions place on the Lagrange multipliers. Recall that if we minimize a function f subject to the constraint $g \le 1$, and if the minimum occurs on the boundary $g = 1$, then we have $\mu_0 \nabla f = \mu_1 \nabla g$ for $\mu_0 \mu_1 \le 0$. This is exactly our situation for $f = J$ and $g = A + B$.

Thus we have three cases: if $\mu_1 = 0$, then equation (38) becomes equation (36) and we've proved the proposition.

In the second case, if $\mu_0 = 0$, then the Euler-Lagrange equation (38) may be solved explicitly for η to find

$$\eta = \tfrac{1}{3} \log(4\|U\|^2) - u_0.$$

Near each pole of U, there is a coordinate z so that $\|U\| = |z|^{-2}$ and $u_0 = \log|\log|z|^2| - \log|z|$. So

$$\eta = \tfrac{1}{3}\log 4 - \tfrac{1}{3}\log|z| - \log|\log|z|^2|$$

there and so $\eta \notin H_1$.

Finally, we consider where $\mu = \mu_1/\mu_0 < 0$. We will analyze the second variation at any critical point to show that there are no minimizers in this case.

Integrate (38) to find

$$-2\pi + 2\pi A + 2\pi B = \mu(-2 \cdot 2\pi A + 2\pi B).$$

Then since $A+B = 1$, we have $2A = B$, since we are in the case $\mu \neq 0$. So $A = 1/3$ and $B = 2/3$. We analyze the second variation to show that for $L < 4/27$, there is no minimizer at $A = 1/3$, $B = 2/3$ (unless possibly if $\mu_1 = 0$).

Let η satisfy (38) and $A = 1/3$, $B = 2/3$. Consider a variation $\eta + \epsilon\alpha + \frac{\epsilon^2}{2}\beta$ so that η satisfies $A+B = 1$ to second order when $\epsilon = 0$.[6]

We assume α is a constant. Then the first variation

$$\frac{\partial}{\partial\epsilon}(A+B)\bigg|_{\epsilon=0} = -2\alpha A + \alpha B = 0$$

for $A = 1/3$, $B = 2/3$. So to first order $\eta + \alpha$ satisfies $A + B = 1$ and α is tangent to $\{A + B = 1\}$.

Now we require

$$\begin{aligned} 0 &= 2\pi \frac{\partial^2}{\partial\epsilon^2}(A+B)\bigg|_{\epsilon=0} \\ &= \int a(4\alpha^2 - 2\beta) + b(\alpha^2 + \beta) \\ &= \alpha^2 2\pi(4A + B) + \int \beta(-2a + b) \\ &= 2\pi \cdot 2\alpha^2 + \int \beta(-2a + b). \end{aligned} \tag{39}$$

[6]This corresponds to an actual variation in Q by standard Implicit Function Theorem arguments—see [17]. Let X be the Banach space $H_1 \cap C^0$. Then let $g : X \to \mathbb{R}$, $g(\nu) = A(\eta + \nu) + B(\eta + \nu)$. It is straightforward to show that g is C^1 in the Banach space sense. Moreover, for $2a \neq b$ (which holds for any $\eta \in H_1$), we can check that $dg : X \to \mathbb{R}$ is nonzero. So then $Y = g^{-1}(1) = \{A + B = 1\}$ is a Banach submanifold of X near $\nu = 0$. So for any element $\alpha \in \ker dg_0$, there is a curve in Y tangent to α. Along such a curve, we compute restrictions on the second-order term β.

Now for this variation $J = J(\eta + \epsilon\alpha + \frac{\epsilon^2}{2}\beta)$, compute

$$\begin{aligned}
\left.\frac{\partial^2 J}{\partial \epsilon^2}\right|_{\epsilon=0} &= \int \nabla\eta\cdot\nabla\beta + |\nabla\alpha|^2 + (2\kappa_0 - \Delta u_0)\beta + \tfrac{3}{2}\int a(4\alpha^2 - 2\beta) \\
&\quad - 2\pi\frac{\int a(4\alpha^2-2\beta) + b(\alpha^2+\beta)}{\int a+b} + 2\pi\frac{\left(\int a(-2\alpha)+b\alpha\right)^2}{\left(\int a+b\right)^2} \\
&= \int[\nabla\eta\cdot\nabla\beta + (2\kappa_0 - \Delta u_0)\beta - 3a\beta] + 2\pi\cdot 6\alpha^2 A \\
&\quad - 2\pi\cdot\alpha^2(4A+B) - \int \beta(-2a+b) \\
&= \int[\nabla\eta\cdot\nabla\beta + (2\kappa_0 - \Delta u_0)\beta - 3a\beta] + 2\pi\cdot 2\alpha^2 \qquad (40)
\end{aligned}$$

Here we've used the following facts to get from the first line to the second: $\nabla\alpha = 0$ since α is constant, $\int(a+b)/2\pi = A + B = 1$, and the last term vanishes since α is constant and $2A = B$. The third line follows from the second by the constraint (39) and the fact $A = 1/3$.

Now we use the Euler-Lagrange equation (38). Recall $\mu_0 \neq 0$ and $\mu = \mu_1/\mu_0$. Then

$$\int \nabla\eta\cdot\nabla\beta = -\int(\Delta\eta)\beta = \int[-(2\kappa_0 - \Delta u_0) + a + b - \mu(-2a+b)]\beta.$$

Plug this into (40) to find

$$\begin{aligned}
\left.\frac{\partial^2 J}{\partial \epsilon^2}\right|_{\epsilon=0} &= (1-\mu)\int(-2a+b)\beta + 2\pi\cdot 2\alpha^2 \\
&= 2\pi\cdot 2\mu\alpha^2.
\end{aligned}$$

Here the last line follows from (39). Thus if we choose $\alpha \neq 0$, then the second variation along this path is negative since $\mu < 0$. Therefore, there is no minimizer for our variational problem satisfying $\mu < 0$. □

Now we show that there is a minimizer.

Lemma 8. *Assume $L < 4/27$. Then there is a constant $\delta > 0$ so that $A, B \in (\delta, 1/\delta)$ for all $\eta \in Q$.*

Proof. Lemma 6 implies that $AB^2 \geq L$. Since $0 < L < 4/27$, $A > 0$, $B > 0$, and $A + B \leq 1$, this proves the lemma. □

Lemma 9. *There are constants K_1, K_2 so that for all admissible $\eta \in H_1$, and for $c = (\int\eta)/(\int 1)$,*

$$\log A \geq K_1 - 2c, \qquad \log B \geq K_2 + c.$$

Proof. Since exp is convex, Jensen's inequality gives

$$\begin{aligned}\log A &= -\log 2\pi + \log \int 4\|U\|^2 e^{-2u_0} e^{-2\eta} \\ &\geq -\log 2\pi + \frac{\int \log 4\|U\|^2 - 2u_0 - 2\eta}{\int 1} + \log \int 1.\end{aligned}$$

The case for B is the same. □

Lemma 10. *Let η_i be a sequence in Q so that $\lim_i J(\eta_i) = \inf_{\eta\in Q} J(\eta)$. Then there is a positive constant C so that $\|\eta_i\|_{H_1} \leq C$ for all i.*

Proof. First we note that Lemmas 8 and 9 show that the average value $c = (\int \eta)/(\int 1)$ is uniformly bounded above and below for all $\eta \in Q$.

Proposition 5 shows that $J(\eta) \geq \gamma \int |\nabla\eta|^2 - R$ for $\gamma, R > 0$ uniform constants. Thus for any minimizing sequence, $\int |\nabla\eta|^2$ must be uniformly bounded. Then write $\eta = \rho + c$ for $\int \rho = 0$, c constant. Then

$$\|\eta\|_{L^2} \leq \|\rho\|_{L^2} + \|c\|_{L^2} \leq \lambda_1^{-\frac{1}{2}} \|\nabla\rho\|_{L_2} + K = \lambda_1^{-\frac{1}{2}} \|\nabla\eta\|_{L_2} + K$$

for K a uniform constant and λ_1 the first nonzero eigenvalue of the Laplacian. This shows the H_1 norm of η in the minimizing sequence is uniformly bounded. □

Now given a minimizing sequence $\{\eta_i\} \subset Q$, Lemma 4 shows that we can assume $A(\eta_i) + B(\eta_i) = 1$, $B(\eta_i) \leq 2/3$. Then there is a subsequence, which we still refer to as η_i, which is weakly convergent to a function $\eta_\infty \in H_1$ (the weak compactness of the unit ball in a Hilbert space), strongly convergent to η_∞ in L^p for $p < \infty$ (Sobolev embedding), convergent pointwise almost everywhere to η_∞ (L^p convergence implies subsequential almost-everywhere convergence), and so that e^{η_i} is strongly convergent to e^{η_∞} in L^p for $p < \infty$ (Moser-Trudinger). Recall

$$J(\eta) = \int [\tfrac{1}{2}|\nabla\eta|^2 + (2\kappa_0 - \Delta u_0)\eta] + 3\pi A - 2\pi \log 2\pi(A+B).$$

Then the second term in the integral converges by strong convergence in L^1 and weak convergence in H_1 (see the proof of Proposition 5 for the integration by parts trick). The term $\int \frac{1}{2}|\nabla\eta|^2$ is lower semicontinuous (the norm in a Hilbert space is lower semicontinuous under weak convergence). Lower semicontinuity is enough since we are seeking a minimizer. B converges by Moser-Trudinger: $e^{u_0} \in L^p$ for $p < 2$. Then since e^{η_i} converges in L^q for $\frac{1}{p} + \frac{1}{q} = 1$, $B = \int 2e^{u_0}e^{\eta}$ converges.

That leaves the term A. Fatou's lemma and the almost-everywhere convergence of η_i then show

$$A(\eta_\infty) \le \liminf_{i\to\infty} A(\eta_i).$$

We want to rule out the case of strict inequality. Note $A(\eta_\infty)+B(\eta_\infty) \le \lim A(\eta_i) + B(\eta_i) = 1$, and so $\eta_\infty \in Q$. Also, since $A(\eta_i) + B(\eta_i) = 1$ and $B(\eta_i) \to B(\eta_\infty)$, $\lim A(\eta_i) = 1 - B(\eta_\infty)$.

Consider the constant k so that $\eta_\infty + k$ minimizes

$$\{J(\eta_\infty + k) : \eta_\infty + k \in Q\}.$$

Note that Lemma 4 shows that $e^{-2k}A(\eta_\infty) + e^k B(\eta_\infty) = 1$. Now compute

$$\begin{aligned} \lim_{i\to\infty} J(\eta_i) &\ge \int \left[\tfrac{1}{2}|\nabla\eta_\infty|^2 + (2\kappa_0 + \Delta u_0)\eta_\infty\right] + 3\pi\left[1 - B(\eta_\infty)\right], \\ J(\eta_\infty + k) &= \int \left[\tfrac{1}{2}|\nabla\eta_\infty|^2 + (2\kappa_0 + \Delta u_0)(\eta_\infty + k)\right] + 3\pi e^{-2k} A(\eta_\infty). \end{aligned}$$

Now substitute $e^{-2k}A(\eta_\infty) = 1 - e^k B(\eta_\infty)$ to show

$$\lim_{i\to\infty} J(\eta_i) - J(\eta_\infty + k) \ge -2\pi k + 3\pi B(\eta_\infty)(e^k - 1) \tag{41}$$

We prove $A(\eta_\infty) = \lim A(\eta_i)$ by contradiction. If on the contrary $A(\eta_\infty) < \lim A(\eta_i)$, Lemma 4 and the fact $A(\eta_i)+B(\eta_i) = 1$ imply that $k < 0$. Then it is straightforward to check that the right-hand side of (41) is strictly positive (it is zero if $k = 0$, and its derivative with respect to k is negative for $k < 0$—use the fact $B(\eta_\infty) = \lim B(\eta_i) \le 2/3$.) This shows $\lim J(\eta_i) > J(\eta_\infty + k)$ and so contradicts the fact that η_i is a minimizing sequence for J.

The same analysis shows that $\lim \int |\nabla\eta_i|^2 = \int |\nabla\eta_\infty|^2$. So $J(\eta_\infty) = \lim J(\eta_i)$, and η_∞ is a minimizer of $\{J(\eta) : \eta \in Q\}$.

Theorem 1. *If $L < 4/27$ then a weak solution to (36) exists. Conversely, if $L \ge 4/27$, then there is no weak solution to (36).*

Proof. The preceding paragraphs, together with Proposition 7, prove existence in the case $L < 4/27$. We address the nonexistence in two cases:

Case $L > 4/27$. If η solves (36), then we can integrate (36) to find $A + B = 1$. On the other hand, $A > 0$, $B > 0$, and Lemma 6 shows that $AB^2 \ge L > 4/27$. Simple calculus shows that there is no such pair (A, B) in this case.

Case $L = 4/27$. As in Case 1, we must have $A + B = 1$ and $AB^2 \ge L = 4/27$. The only way this can happen is if $A = 1/3$, $B = 2/3$, so that $AB^2 = 4/27$. In this case, Lemma 6 forces $\eta = C + \frac{2}{3}\log\|U\| - u_0$ for some constant C. Since $u_0 = \log|\log|z|^2| - \log|z|$ and $\|U\| = |z|^{-2}$ near each pole of U, $\eta = C - \log|\log|z|^2| - \frac{1}{3}\log|z|$ near each pole of U. Thus $\eta \notin H_1$. □

Proposition 11. *Any weak solution η to (36) is smooth away from the poles of U.*

Proof. In a neighborhood bounded away from the poles of U, the quantities $\|U\|^2$ and u_0 are smooth and bounded. Since $\eta \in H_1$, Moser-Trudinger shows that e^η, $e^{-2\eta} \in L^p$ for all $p < \infty$. Therefore, (36) implies $\Delta\eta \in L^p_{\text{loc}}$. Since $\eta \in L^p$ by Sobolev embedding, the L^p elliptic theory [10] shows that $\eta \in W^{2,p}_{\text{loc}}$. Sobolev embedding shows $\eta \in C^{0,\alpha}_{\text{loc}}$, and so $\Delta\eta \in C^{0,\alpha}_{\text{loc}}$. The Schauder theory then shows $\eta \in C^{2,\alpha}_{\text{loc}}$. Further bootstrapping implies η is smooth. □

7.3. **A metric for the "Y" vertex.** Let $\widetilde{\Sigma}$ be the universal cover of $\Sigma = S^2 \setminus \{p_1, p_2, p_3\}$. Lifting the appropriate objects to the cover we find a solution to (32) on $\widetilde{\Sigma}$. Since the equation (32) is the integrability condition for the developing map, we have a solution $\widetilde{f} : \widetilde{\Sigma} \to \mathbb{R}^3$, with monodromies of Σ acting as equiaffine deck transformations fixing the normal vector ξ and acting by isometry. The quotient by the deck transformations gives an elliptic affine sphere structure on Σ as well as the locally defined developing map f. Then the map $F : (\Sigma \times \mathbb{R}_+) \to \mathbb{R}^3$ defined by $F(x, r) = rf(x) =: (y_1, y_2, y_3)$ maps the cone over Σ to $\mathbb{R}^3$ and is locally invertible (so we may express $r = r(y)$). The potential function $\Phi(y) = r^2/2$ defines a parabolic affine sphere on a neighborhood of the "Y" vertex, by the result of Baues and Cortés (see Example 10). This is our main result.

Remark. The monodromy group of this metric determines the affine flat structure. We have not yet determined this monodromy group, thus cannot verify that the metric is one predicted by Gross and Siebert [12].

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