RESEARCH STATEMENT (revised 10/01/08)

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1. Overview

My work focuses on the geometry and differential equations invariant under groups of affine and projective motions (in $\mathbb{R}^n$ and $\mathbb{RP}^n$ respectively). In particular, affine differential geometry, the study of properties of hypersurfaces in $\mathbb{R}^{n+1}$ which are invariant under affine volume-preserving motions, informs most of my work. Affine differential geometry is an old subfield of geometry, with Blaschke making much progress in the 1920s, but it is intimately related with many other topics of current interest: real Monge-Ampère equations, mirror symmetry of Calabi-Yau manifolds, and $\mathbb{RP}^2$ structures on surfaces. My research primarily consists of exploring these relationships.

A slightly different point of view of my research is to consider manifolds with canonical coordinate charts—(X, G) structures in Thurston's parlance, with coordinate charts in $X = \mathbb{R}^n$ or $\mathbb{RP}^n$ and gluing maps in G the affine or projective group, respectively. The rigidity of this construction allows for more interesting geometric and analytic structure on such a manifold. For example, the Hessian of a smooth function is a tensor on an affine manifold, and its determinant (the real Monge-Ampère operator) also makes sense. The model here is complex geometry: On a complex manifold (with holomorphic coordinate gluing maps), one defines Kähler metrics locally as complex Hessians of potential functions, and then natural complex Monge-Ampère equations lead to the rich geometry of Kähler-Einstein metrics. Much of my work involves similar PDEs on affine and projective manifolds, with the successes of complex geometry as a model.

2. $\mathbb{RP}^n$ Structures

An $\mathbb{RP}^n$ structure on an n-dimensional manifold is given by an atlas of coordinate charts in $\mathbb{RP}^n$ with gluing maps in $\mathbf{PGL}(n+1, \mathbb{R})$. The Klein model of hyperbolic space shows that all hyperbolic structures induce an $\mathbb{RP}^n$ structure. An $\mathbb{RP}^n$ manifold M is *properly convex* if we can write $M = \Omega/\Gamma$, where $\Omega \Subset \mathbb{R}^n \subset \mathbb{RP}^n$ is a convex domain and $\Gamma \subset \mathbf{PGL}(n+1, \mathbb{R})$ acts discretely and properly discontinuously. The domain Ω may be identified canonically with a convex hypersurface H, the hyperbolic affine sphere, which is asymptotic to the cone over Ω in $\mathbb{R}^{n+1}$ and is invariant by any volume-preserving linear actions on the cone. The existence of H follows from a Monge-Ampère equation solved by Cheng-Yau.

In the case of convex $\mathbb{RP}^2$ surfaces, Goldman uses an analog of the Fenchel-Nielsel coordinates on Teichmüller space show the deformation space of convex $\mathbb{RP}^2$ structures on a closed oriented surface of genus $g \geq 2$ is homeomorphic to a cell of dimension $16g-16$. A natural problem is then to extend the rich structures on Teichmüller space to this larger deformation space. In [1][1], I provide a mapping-class-invariant complex structure on this deformation space by showing that a properly convex $\mathbb{RP}^2$ structure on a closed oriented surface of genus $g \geq 2$ is equivalent to a pair (Σ, U) of a conformal structure Σ and a holomorphic cubic differential U (this is also due independently to François Labourie). The complex structure on the deformation space is then induced by viewing it as the total space of a holomorphic vector bundle over Teichmüller space with fibers the space of cubic differentials. The proof involves work of Cheng-Yau and C.P. Wang on hyperbolic affine spheres.

The moduli space of Riemann surfaces of genus $g \geq 2$ is the quotient of Teichmüller space by the mapping class group. The Deligne-Mumford compactification of the moduli space involves adding stable nodal Riemann surfaces, in which nontrivial necks of the surface are pinched to singular nodal intersections. In terms of hyperbolic geometry, the neck pinch makes the hyperbolic circumference of the neck shrink to zero, which implies the corresponding Fuchsian group element degenerates from hyperbolic to parabolic.

In [4], I study limits analogous to neck pinches in moduli space of convex $\mathbb{RP}^2$ structures. Thus we consider limits of pairs $(\Sigma_t, U_t) \to (\Sigma_\infty, U_\infty)$, where Σ_∞ is a nodal (punctured) Riemann surface at the Deligne-Mumford boundary of moduli space, and U_∞ is a regular cubic differential on the Σ_∞—i.e., U_∞ has poles of order up to three at the punctures, and its residues (dz^3/z^3 coefficients) sum to zero at identified puncture points.

In [4], I identify the $\mathbb{RP}^2$ holonomy type at each puncture of Σ_∞ explicitly in terms of the residue of U_∞, and also observed the limit of the holonomy in almost all degenerating families $(\Sigma_t, U_t) \to (\Sigma_\infty, U_\infty)$. In particular, all holonomy types in $\mathbf{SL}(3,\mathbb{R})$ with positive real eigenvalues and maximal Jordan blocks (when there are repeated eigenvalues) are represented by such limits of compact $\mathbb{RP}^2$ structures. So, in contrast to the case of hyperbolic Riemann surfaces, there are limiting families of convex $\mathbb{RP}^2$ surfaces for which the holonomy remains hyperbolic (positive distinct eigenvalues) and does not degenerate to a parabolic

[1]The numbers [∗] refer to my vita.

holonomy. Instead, the holonomy can remain fixed while a generalized "twist" parameter introduced by Goldman blows up in the limit.

The proof involves solving the equation (of Wang) to produce a conformal factor for the affine metric on a hyperbolic affine sphere:

$$\Delta u + 4\|U\|^2 e^{-2u} - 2e^u - 2\kappa = 0, \tag{1}$$

on the punctured Riemann surface Σ_∞, and the Laplacian Δ and curvature κ are with respect to a background conformal metric on Σ_∞. This equation behaves very well with respect to the maximum principle, and we can construct good barriers to control u at the punctures.

In [7], I study limits of $\mathbb{RP}^2$ structures with conformal structure fixed and cubic differential going to infinity: (Σ, tU_0) as $t \to \infty$, following work of Anne Parreau and Inkang Kim. The limiting strucure involves the asymptotics of the eigenvalues of the holonomy along free loops in Σ (this generalizes Thurston's boundary of Teichmüller space). Such limits usually require nonstandard analysis (nonprincipal ultrafilters). For (Σ, tU_0) as $t \to \infty$, I determine the limiting holonomy along many loops in terms of the geometry of the flat metric $|U_0|^{\frac{2}{3}}$, without using nonstandard analysis. I am currently working to extend this result to determine the limiting holonomy along all free loops in Σ, and to determine whether nonstandard analysis is needed for such limits.

On closed Riemann surfaces of genus at least 2, Labourie identifies C.P. Wang's cubic differential with a cubic differential given by Hitchin in his study of representations of higher-genus surface groups into $\mathbf{SL}(n, \mathbb{R})$ for $n = 3$. Bill Goldman, Anna Wienhard and I are studying the geometry of Hitchin representations into $\mathbf{Sp}(4, \mathbb{R})$, which involve quartic differentials, not cubic differentials, on a Riemann surface of genus at least 2.

Schoen-Yau prove that any conformally flat manifold M which admits a conformal Riemannian metric of positive scalar curvature is the conformal quotient of a domain $\Omega \subset \mathbb{S}^n$, where $\mathbb{S}^n \setminus \Omega$ has small Hausdorff dimension. In other words, Schoen-Yau show that the developing map from the universal cover of M to conformal $\mathbb{S}^n$ is injective. In [2,3], I establish an analogous result for projective manifolds: A closed $\mathbb{RP}^n$ manifold M is properly convex (i.e., the developing map is bijective to a convex domain) if and only if M admits a Riemannian metric of the form $-\frac{1}{u}u_{ij}$ for u a convex negative section of the tautological bundle.

3. Affine and Complex Manifolds

An affine manifold is a manifold equipped with coordinate charts in $\mathbb{R}^n$ with constant gluing maps in the affine group $\mathrm{Aff}(n, \mathbb{R})$. Alternately, a flat affine connection D on the tangent bundle of a smooth manifold provides the structure of an affine manifold. The tangent bundle of an affine manifold admits a natural complex structure. A D-invariant volume form on an affine manifold induces a holomorphic volume form on the tangent bundle, and in this case a Riemannian metric g_{ij} locally given by the Hessian of a convex potential function ϕ induces a Ricci-flat Kähler metric on the tangent bundle if the real Monge-Ampère equation $\det \frac{\partial^2 \phi}{\partial x^i \partial x^j} = 1$ is satisfied. If the linear part of the affine holonomy is integral (gluing maps in $\mathbf{SL}(n,\mathbb{Z}) \ltimes \mathbb{R}^n$), there is a D-invariant lattice bundle contained in the tangent bundle, and so there is a natural torus bundle.

The famous conjecture of Strominger-Yau-Zaslow addresses Calabi-Yau manifolds at a very singular limit of the moduli space (the large complex structure limit): Such a Calabi-Yau manifold should be the total space of a fibration with base B and generic fibers special Lagrangian tori. The base B should contain a singular locus S of real codimension 2, and $B \backslash S$ should be an integral affine manifold equipped with a semi-flat Calabi-Yau metric. The special Lagrangian torus fibers are then quotients of the tangent spaces of points in $B \setminus S$. This conjecture provides a geometric framework for the mirror pairing of two Calabi-Yau manifolds predicted by string theory, the mirror, in this limiting case, being constructed in part from the dual semi-flat Calabi-Yau structure on $B \setminus S$ given by the Legendre transform of the Calabi-Yau potential.

In [5] I find many examples of semi-flat Calabi-Yau manifolds in dimension 2. Given any meromorphic cubic differential U on $\mathbb{CP}^1$ with only simple poles, I produce an affine structure and semi-flat Calabi-Yau metric on $\mathbb{CP}^1$ minus the pole set of U. The affine holonomy type around each singularity (pole) is verified to be the same as the limits of Calabi-Yau metrics on K3 surfaces constructed Gross-Wilson. The proof involves solving

$$\Delta u + 4\|U\|^2 e^{-2u} - 2\kappa = 0, \tag{2}$$

on $\mathbb{CP}^1$ minus the poles of U, so that e^u is the conformal factor for the semi-flat Calabi-Yau metric (or equivalently, the metric for a parabolic affine sphere). The maximum principle can be applied as in our solution to (1) above, but with a little more difficulty.

In [6,6e], Yau, Zaslow and I produce non-trivial semi-flat Calabi-Yau metrics in dimension 3. In dimension 3, the singular set S of the base B should be a graph, generically with trivalent vertices. Yau, Zaslow and I construct many semi-flat Calabi-Yau metrics on a neighborhood the complement of a trivalent-vertex graph in $\mathbb{R}^3$. We have two constructions to produce these examples. First, a result of Baues-Cortés shows that an elliptic affine sphere in dimension two leads to a solution to the Monge-Ampère equation $\det \frac{\partial^2\phi}{\partial x^i \partial x^j} = 1$ in dimension three. The structure of an elliptic affine sphere follows from solving

$$\Delta u + 4\|U\|^2 e^{-2u} + 2e^u - 2\kappa = 0. \tag{3}$$

on $\mathbb{CP}^1$, for U a sufficiently small meromorphic cubic differential on $\mathbb{CP}^1$ with exactly three poles of order 2 [6,6e]. Our second construction is to extend the result of Baues-Cortés to show that a hyperbolic affine sphere in dimension two also leads to a solution of the Monge-Ampère equation in dimension three. The equation in this case is (1) above, and by [4], we can solve this (1) for U any meromorphic cubic differential on $\mathbb{CP}^1$ with exactly three poles of order ≤ 3.

In each of these two cases, since U has three singularities, the cone over $\mathbb{S}^2$ minus the poles of U is diffeomorphic to a ball in $\mathbb{R}^3$ minus the "Y" vertex of a graph.

Although the existence problem for Kähler-Einstein metrics on compact Kähler manifolds is still unsolved, the analogous problem for holomorphic vector bundles on such manifolds has a complete solution: Donaldson and Uhlenbeck-Yau prove that an irreducible holomorphic vector bundle over a closed Kähler manifold admits a Hermitian-Einstein metric if and only if the bundle is slope-stable (this is often called the Kobayashi-Hitchin correspondence). Li-Yau extend this result to non-Kähler Gauduchon complex manifolds. Among the important implications of this result are Li-Yau-Zheng's and Teleman's work to help classify complex surfaces, and the first step of Li-Yau's solution of Strominger's equations from string theory on non-Kähler manifolds.

In [9], I extend Donaldson-Uhlenbeck-Yau's theory to compact affine manifolds with affine invariant volume form (i.e., to special affine manifolds, since the affine holonomy lies in $\mathbf{SL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$). Along the way, I adapt the theory of Gauduchon metrics to special affine manifolds and slope stability to flat vector bundles over special affine manifolds. The main theorem is this: A flat vector bundle on a special affine manifold either has a destabilizing flat subbundle (with respect to a background Gauduchon metric) or admits an affine Hermitian-Einstein bundle metric. Note that the destabilizing subbundle is smooth, and we do not have to resort to singular subsheaves to define stability as in the complex case.

I expect to apply the theory of affine Hermitian-Einstein metrics to study special affine manifolds of low dimension (following the work of

Li-Yau-Zheng and Teleman on complex surfaces). Also, affine Hermitian-Einstein metrics thus provide a first step to understanding "real slices" of Li-Yau's examples, much as semi-flat Calabi-Yau metrics on affine manifolds form a large part of Strominger-Yau-Zaslow's conjecture.

4. Affine Normal Flow

The affine normal flow is a parabolic flow of convex hypersurfaces in $\mathbb{R}^{n+1}$ which is invariant under volume-preserving affine transformations and is equivalent to the flow by $K^{\frac{1}{n+2}}\nu$ for K the Gauss curvature and ν the Euclidean normal. All the flows of the form $K^\alpha \nu$ are equivalent to parabolic Monge-Ampère equations for the support function. For compact initial hypersurfaces, Ben Andrews proves that the rescaled limit of the affine normal flow at its extinction time is an ellipsoid.

In [8,10], Mao-Pei Tsui and I define the affine normal normal flow for all initial convex noncompact hypersurfaces in $\mathbb{R}^{n+1}$ which are properly embedded and which contain no lines. Our work provides the first example of a power-of-Gauss curvature flow to be extended to the noncompact case.

- Ancient Solutions: We classify all ancient solutions of the affine normal flow as either ellipsoids (which shrink to a point), or paraboloids (which translate in space).
- Instantaneous Smoothing: As expected from Andrews's work in the compact case, all solutions are smooth and strictly convex for positive time.
- Long-Time Existence: The affine normal flow of any initial noncompact hypersurface, as above, exists for all positive time.
- Maximum Principle at Infinity: If an initial hypersurface $\mathcal{L}$ lies inside another initial hypersurface $\mathcal{L}'$, then this property is preserved under the affine normal flow.
- Expanding Solitons: We recover Cheng-Yau's existence of hyperbolic affine spheres asymptotic to any convex cone containing no line (given existence, the asymptotic property is due to Sasaki and Gigena).
- Dirichlet Problem: By looking at the support function of the evolving hypersurface, we are able to solve a Dirichlet problem for the parabolic Monge-Ampère equation $s_t = -(\det s_{ij})^{-\frac{1}{n+2}}$ on a bounded convex domain $\Omega \subset \mathbb{R}^n$, for any initial $s_0 \colon \mathbb{R}^n \to (-\infty, +\infty]$ which is convex in the sense that the region above its graph is closed and convex in $\mathbb{R}^{n+1}$. (Here Ω is the interior of $s_0^{-1}(-\infty, \infty)$.) This includes the possibility that $s_0|_{\partial\Omega}$ is discontinuous or infinite, and that s_0 is only lower semicontinuous

approaching $\partial\Omega$ from the interior. This Dirichlet problem for finite, discontinuous boundary values seems to be a new phenomenon among Monge-Ampère equations.

The methods Tsui and I use to study the affine normal flow include estimates of Andrews and Gutiérrez-Huang and barriers due to Calabi. Tsui and I are working on analyzing the long-time behavior of noncompact solutions to the affine normal flow.

5. Graduate Real Analysis Course

While teaching at Rutgers-Newark, I have developed 164 pages of notes, available on my website, for a graduate course in Real Analysis II. The material focuses on developing analytic techniques for producing a length-minimizing curve in a free homotopy class of a closed Riemannian submanifold of $\mathbb{R}^N$, developing the necessary theory of ODEs, the calculus of variations, distributions, Hilbert spaces, and the Ascoli-Arzelá Theorem along the way. The idea is to introduce students quickly to many of the important analytic tools necessary in geometric analysis, without devoting too much time to abstract functional analysis.