1. Affine Spheres

As part of their great burst of activity in the late 1970s, Cheng and Yau proved many geometric results concerning differential structures invariant under affine transformations of $\mathbb{R}^n$.

Affine differential geometry is the study of those differential properties of hypersurfaces in $\mathbb{R}^{n+1}$ which are invariant under volume-preserving affine transformations. One way to develop this theory is to start with the affine normal, which is an affine-invariant transverse vector field to a convex $C^3$ hypersurface. A hypersurface is an affine sphere if the lines formed by the affine normals all meet at a point, called the center. A convex affine sphere is called elliptic, parabolic, or hyperbolic according to whether the affine normals point toward the center, are parallel (the center being at infinity), or away from the center, respectively.

The global theory of elliptic and parabolic affine spheres is quite tame: Every properly immersed elliptic affine sphere is an ellipsoid, while every properly immersed parabolic affine sphere is a paraboloid. In this generality, both these results follow from Cheng-Yau’s paper [8], in which they show that any properly immersed affine sphere must have complete affine metric. Then one may appeal to earlier results of Calabi [3] to classify global elliptic and parabolic affine spheres. The global classification of parabolic affine spheres is an extension of the well-known result of Jörgens, Calabi, and Pogorelov that any entire convex solution to the Monge-Ampère equation $\det u_{ij} = 1$ on $\mathbb{R}^n$ is a quadratic polynomial.

Calabi realized that hyperbolic affine spheres are more varied, by noting that two quite different convex cones contain hyperbolic affine spheres asymptotic to their boundaries. In addition to the hyperboloid, asymptotic to a round cone over an ellipsoid, Calabi also wrote down an affine sphere asymptotic to the boundary of the first orthant in $\mathbb{R}^{n+1}$, which is a cone over an $n$-dimensional simplex [3]. Based on these explicit examples in these two extremal cases of convex cones, Calabi conjectured that each proper convex cone admits a unique (up
to scaling) hyperbolic affine sphere, and that every properly immersed hyperbolic affine sphere in $\mathbb{R}^{n+1}$ is asymptotic to the boundary of such a cone. Moreover, he conjectured that the proper immersion of a hyperbolic affine sphere is equivalent to the completeness of an intrinsic affine (or Blaschke) metric.

Cheng-Yau prove Calabi’s conjecture on hyperbolic affine spheres in [8, 6]. Calabi-Nirenberg proved the same results as in [8] around the same time in unpublished work. One of the main techniques in Cheng-Yau’s proof is a gradient estimate on a height function. (There are also clarifications of Cheng-Yau’s proof in Li [11, 12].)

2. Hyperbolic Affine Spheres and Real Monge-Ampère Equations

If $\Omega \subset \mathbb{R}^n \subset \mathbb{R}P^n$ is a convex domain, then the existence of a hyperbolic affine sphere asymptotic to the cone over $\Omega$ follows from the solution of the following Dirichlet problem for a real Monge-Ampère equation

\[
\det u_{ij} = \left( -\frac{1}{u} \right)^{n+2}, \quad (u_{ij}) > 0, \quad u \big|_{\partial \Omega} = 0.
\]

Calabi conjectured that there is a unique solution to (1) on any convex bounded $\Omega$ [3]. Cheng-Yau show there always exists such a solution in [6], and uniqueness follows easily by the maximum principle. From this solution, one may use a duality result of Calabi (known to experts at the time of Cheng-Yau’s work and published later in [9]), or a later argument of Sasaki [15], to produce the hyperbolic affine sphere asymptotic to the cone over $\Omega$.

The paper [6] was one of the first works to prove the existence smooth solutions to general real Monge-Ampère equations on convex domains. The technique is to prove regularity of Alexandrov’s weak solution. A key step in the proof is to approximate solutions to the Dirichlet problem for a real Monge-Ampère equation by solutions to the Minkowski problem on $\mathbb{S}^n$, which are provided by Cheng-Yau in [5].

Loewner-Nirenberg solved (1) earlier in the case of domains in $\mathbb{R}^2$ with smooth boundary [13]. Cheng-Yau’s result requires no regularity of $\partial \Omega$ except that provided by convexity. The solution of (1) for $\partial \Omega$ only Lipschitz relies on using Calabi’s explicit solution on a simplex as a barrier. The case of rough boundary is of particular geometric interest, as most convex domains admitting cocompact projective group actions have boundaries which are nowhere $C^2$ [10, 1].
Cheng-Yau provide another existence and regularity proof for the real Monge-Ampère equation in [7], this time using a tube domain construction to gain access to Yau’s estimates for the complex Monge-Ampère equation [17].

3. AFFINE MANIFOLDS

An affine manifold is a manifold with coordinate charts in $\mathbb{R}^n$ and affine gluing maps $x \mapsto Ax + b$. The tangent bundle to an affine manifold carries a natural complex structure, which can be provided by gluing together tube domains over the affine coordinate charts in $\mathbb{R}^n$. There is also a natural notion of an affine Kähler, or Hessian, metric, which is the natural restriction of a Kähler metric on the total space of the tangent bundle which is invariant in the bundle directions.

In [7], Cheng-Yau asked and answered the analogous question for Yau’s solution to the Calabi conjecture on Kähler manifolds—the existence of Kähler-Einstein metrics. In the Kähler case, any closed Kähler manifold with first Chern class $c_1 < 0$ admits a unique Kähler-Einstein metric of negative Ricci curvature. Cheng-Yau define an affine first Chern class and prove the negativity of this affine Chern class on a closed affine Kähler manifold is equivalent to the existence of a complete Kähler-Einstein metric on the tangent bundle. The restriction of this metric to the affine manifold is nowadays called the Cheng-Yau metric. The class of affine manifolds admitting a complete Cheng-Yau metric is exactly the class of affine quotients of proper convex cones. Moreover, the Cheng-Yau metric on a convex cone and the hyperbolic affine sphere asymptotic to the cone are equivalent [16, 14]. (Note the Cheng-Yau metric is not in general Einstein: it is merely the restriction of the Kähler-Einstein metric on the tangent bundle.)

The affine analog of a Calabi-Yau manifold (a closed Kähler manifold with $c_1 = 0$) is an affine Kähler manifold admitting a parallel volume form. On any closed affine Kähler manifold with parallel volume, Cheng-Yau construct a flat affine-Kähler metric by appealing to estimates in [17].

4. MAXIMAL HYPERSURFACES IN MINKOWSKI SPACE

Calabi conjectured in [2] that every entire maximal spacelike hypersurface in Minkowski space $\mathbb{R}^{n,1}$ must be a hyperplane, and provided a proof for $n \leq 4$. In [4], Cheng-Yau prove this result for all $n$, using similar estimates to those in [8]. This Bernstein property for maximal spacelike hypersurfaces is in contrast to the case of entire minimal
graphs in $\mathbb{R}^{n+1}$. These must be linear for $n < 8$, but there are entire nonlinear examples for $n \geq 8$.

**References**


