To the beginning Calculus student,

We may often think that just as we have the “golden rule” for equations, so too do we have it for inequalities. For the neophytic Calculus student relying on algebraic manipulation rather than on geometrical interpretation, this line of thinking is an easy trap to fall into. Plus it is never really discussed in class. So while inequalities may hold for some operations, it does not work in general for differentiation. This first occurred to me when I was solving problems in Berkeley Problems in Mathematics, and looking for differential/integral equations. I came across the problem: Let $f$ be a real valued continuous nonnegative function on $[0,1]$ such that

$$f(t)^2 \leq 1 + 2 \int_0^t f(s) ds$$

for $t \in [0,1]$. Show that $f(t) \leq 1 + t$ for $t \in [0,1]$.

Prior to looking at this book I had done two problems that seemed “almost” exactly the same; one of which is from Stewart’s Calculus textbook: Find all functions $f$ such that $f'(x)$ is continuous and

$$f(x)^2 = 100 + \int_0^x [f(t)^2 + (f'(t))^2] dt$$

for all real $x$. 

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The other is problem number 562 in Putnam and Beyond: Find all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) that satisfy

\[
f(x) + \int_0^x (x-t)f(t)dt = 1, \text{ for all } x \in \mathbb{R}.
\]

After solving these two problems it was not hard to think that I should use the same methodology for the first one and so accordingly, I blindly "solved" the Berkeley problem by the usual differentiating of both sides, and solving the resulting ODE and in fact got the correct answer. Nevertheless, when I looked at the book’s solution this is not how it was done. First off, you are probably wondering why I was allowed to differentiate either of the two equations in the first place. Stewart’s problem is pretty obvious since we are given that \( f \) is differentiable and so we need only show that \( f^2 \) is differentiable either by the chain rule or straight from the definition. Thus the right hand side must be differentiable as well since it is equal to a differentiable function. However, less obvious is the second equality since we are given that \( f \) is only continuous; have no fear, the Fundamental Theorem of Calculus (FTC) comes to the rescue!

Although, with the inequality you got me; I was wrong to do that since the Fundamental Theorem of Calculus does not apply in this case. But suppose the problem has the stronger condition that \( f \) is differentiable; then this is just an accident waiting to happen because it gives the green light to just go ahead and differentiate as one would feel justified in doing so! Alas, this would be wrong as I finally found that in general

If \( f \) and \( g \) are differentiable functions on an interval \( I \subseteq \mathbb{R} \) and \( f \leq g \), this DOES NOT imply \( f' \leq g' \) on \( I \).

Probably the simplest counterexample uses constant functions: \( x^2 \leq 25 \) on \([1,5]\), but differentiating yields \( 2x \leq 0 \) which is clearly not true on \([1,5]\). Let’s do something a little more substantial: on the interval \([0,1]\) we have

\[
e^{-x^2} + 1 \geq x^2 + e^{-1}
\]

which is fairly obvious if we look at the graphs of the two functions. Nevertheless, once we take the derivative

\[
-2xe^{-x^2} \geq 2x \Rightarrow -e^{-x^2} > 1 \text{ for } x > 0
\]

we see immediately that the resulting inequality is wrong on \([0,1]\). In fact, not only is the inequality wrong because of the negative sign, but the maximum value of \( e^{-x^2} \) is 1 on \([0,1]\), so it cannot be greater than one.
Actually this notion of not being able to differentiate inequalities is quite obvious by looking at an illustration of the general case (Fig.1 (a)). The picture shows the reason differentiation does not preserve inequalities, simply put we are now comparing two totally different things. In the first place the inequality compares the distance above the $x$-axis for each function on some interval, but once we differentiate we are now comparing the slope of the tangent line at each point along the same interval. Moreover, the slope has to do with the structure and geometry of the function, not its translation on the $xy$-plane. Even though this idea seems obvious, it is applicable to many problems.

Alternatively, we can turn it into a physics problem: just because two people run for some stretch of time and one is always ahead of the other does not mean that the person ahead runs faster in that interval of time (since velocity is the derivative of position); the one in the lead just happened to be far enough ahead initially. On the other hand, if one person is running faster than another person, then the faster runner must go a greater distance if they both run for the same amount of time on the same terrain (position or distance being the integral of velocity).

So integration preserves inequalities, even though we are still comparing different things! However this time the height above the $x$-axis is in direct correlation with the integral of a function (namely the area under the curve). Again this can also clearly be seen by a simple geometrical picture (Fig.1 (b) 1.2). Therefore, even though certain operations do not preserve inequalities, that does not mean that their inverse operations do not. Consequently, this makes a nice application to Calculus II since the comparison tests are constantly being used to show the convergence for both infinite series and improper integrals. This can be done by differentiating inequalities for simple functions and reaching an obviously true inequality; then we can just integrate back. Simple functions are needed because if they are not simple, taking the derivative will only complicate expressions instead of “breaking them down.” Here is an example.

We want to show that the improper integral (problem from [3])

$$\int_1^\infty \frac{dx}{x + e^{2x}}$$

converges using the comparison test. Now it seems fairly obvious that $x + e^{2x} \geq e^x$ on $[1, +\infty)$, and so we can just flip the expressions and the inequality sign and $\int_1^\infty e^{-x}dx$ converges which finishes the problem. However, we can do better; we can get a four for one deal with just a little more work. Differentiating the previous inequality twice we obtain

$$4e^{2x} \geq e^x \Rightarrow 4e^x \geq 1$$

which is obviously true (actually on the interval $[0, +\infty)$). Since we are starting from an
Figure 1:
inequality we know to be true, now we can integrate back

\[4e^x \geq 1 \Rightarrow 4e^{2x} \geq e^x \Rightarrow 4 \int_0^x e^{2t} \, dt \geq \int_0^x e^t \, dt\]

\[\Rightarrow 2e^{2x} - 1 \geq e^x\]

\[\Rightarrow \int_0^x (2e^{2t} - 1) \, dt \geq \int_0^x e^t \, dt \Rightarrow e^{2x} - x \geq e^x\]

which is a stronger inequality and proves that

\[\int_0^\infty \frac{dx}{x + e^{2x}} \text{ and } \int_0^\infty \frac{dx}{e^{2x} - x} \, dx\]

converge, the second of which may not be obvious using the usual tools for the comparison test. Moreover, this shows that the infinite sums

\[\sum_{n=0}^\infty \frac{1}{n + e^{2n}} \text{ and } \sum_{n=0}^\infty \frac{1}{e^{2n} - n}\]

converge either by the comparison test or the integral test.

Here is another problem from Putnam and Beyond # 404, show that for

\[x \in \mathbb{R} \setminus 0, e^x > x + 1.\]

How can we use this same method to solve this problem?

I have saved the proof of the Berkeley problem till the end to allow time for the reader to contemplate how to really do it. Since now we know we can integrate inequalities we would like to generate some derivatives. Let \(G(t) = \int_0^t f(s) \, ds \Rightarrow G'(t) = f(t)\) by the FTC since \(f\) is continuous. Taking the positive square root, dividing by this term and tacking on a \(\frac{1}{2}\) we obtain:

\[\frac{f(t)}{2\sqrt{1 + 2G(t)}} \leq 1.\]

The first term is just the derivative of \(\sqrt{1 + 2G(t)}\), so

\[\int_0^t \frac{d}{dy}(\sqrt{1 + 2G(y)}) \, dy \leq \int_0^t dy \Rightarrow \sqrt{1 + 2G(t)} \leq t + 1\]
which completes the problem.

A question may come to mind now: under what conditions can we differentiate an inequality? Here is a small proposition, and later we give necessary and sufficient conditions.

**Definition:** The distance between two curves, $f$ and $g$, at a point $x$ is the vertical distance $d(x) = |f(x) - g(x)|$.

**Proposition 1:** If $f$ and $g$ are differentiable functions with $f(x) \leq g(x)$ on $[a,b] = I \subseteq \mathbb{R}$ and the distance between the curves is increasing or constant for all distinct points $x \in I$, then $f' \leq g'$ on $I$. Otherwise, if the distance is decreasing or constant, then $f' \geq g'$.

**Proof.** Suppose $f(x) \leq g(x)$ on $I$, and that the distance between the curves is increasing or constant. Then for $x_1, x_2 \in I$ and $d(x_1) = |f(x_1) - g(x_1)|$, $d(x_2) = |f(x_2) - g(x_2)|$, we have for

$$x_1 < x_2 \Rightarrow d(x_1) \leq d(x_2).$$

So

$$|f(x_1) - g(x_1)| \leq |f(x_2) - g(x_2)| \Rightarrow f(x_2) - f(x_1) \leq g(x_2) - g(x_1).$$

Hence

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{g(x_2) - g(x_1)}{x_2 - x_1}$$
and the desired result follows from taking the limit as $x_2 \to x_1$.

If the distance is decreasing or constant, just reverse the inequality and use a similar argument to obtain the reverse inequality.

With two more conditions on $f(x)$ in the Berkeley problem: differentiability and it being decreasing, it turns out that we are actually validated in differentiating the inequality by Proposition 1. It just has to be shown that that

$$t_1 < t_2 \Rightarrow 1 - f(t_1)^2 + 2 \int_0^{t_1} f(s)ds \leq 1 - f(t_2)^2 + 2 \int_0^{t_2} f(s)ds.$$

For $t_1 < t_2 \Rightarrow \int_0^{t_1} f(s)ds \leq \int_0^{t_2} f(s)ds$ since $f$ is nonnegative on the interval $[0,1]$ so the area is increasing (equality only holds if $f \equiv 0$). So assume $f \neq 0$. Then since $f(t_1) \geq f(t_2)$
\[ t_1 < t_2 \Rightarrow 1 - f(t_1)^2 + 2 \int_{0}^{t_1} f(s) ds \leq 1 - f(t_2)^2 + 2 \int_{0}^{t_2} f(s) ds. \]

as desired.

Now we can differentiate the original inequality obtaining

\[ 2f f' \leq 2f \Rightarrow f(f' - 1) \leq 0. \]

\[ \Rightarrow \text{either } f \leq 0 \text{ and } f' - 1 \geq 0. \]

or

\[ f \geq 0 \text{ and } f' - 1 \leq 0. \]

The latter must be true since \( f \) is not negative and not identically zero. Integrating the right inequality obtains the correct answer (if something does not seem obvious, look back at the original inequality of the problem).

Maybe we also want to solve integro-differential inequalities of a similar form:

\[ F(f(t), f', ..., f^{(n-1)}, t) \leq \int_{a}^{t} Y(f(s), ..., f^{(n-1)}, f^{(n)}) G(s) ds, \text{ for } t \in A \subseteq \mathbb{R}. \]

If the integrand is non-negative (or non-positive) and the function on the left of the inequality is decreasing on the interval, then this inequality can be differentiated, which may allow an easier method to solve it (get the best bound for the function \( f(t) \)). However, these conditions may be rather restrictive since we have to know how higher derivatives of the function behave. Here is another example to clarify this:

\[ [f''(t)]^2 \leq 5 + 2 \int_{0}^{t} f'''(s)f(s)ds \]

If \( f \) and its 3rd derivative are both nonnegative and \( f'' \) is decreasing, then we can differentiate and get

\[ 2f'' f''' \leq 2f''' f \Rightarrow f'''(f'' - f) \leq 0, \]

which is easier to solve.

So when the conditions are met we can turn integral inequalities into differential inequalities by differentiating. Although, once we have these ordinary differential inequalities (ODIs) we would like to solve them in analogy with ODEs. For 1st order linear and separable ODIs, the analogy works since the process only involves integration and multiplication of functions.
(although as we shall see, not with total satisfaction). However for 2nd order and higher ODEs, we need a lot more initial information than than would be required for just equality.

**Example 1.** Suppose that differentiation of an equality is valid, and it yields the ODI

\[ f'(t) \leq 2t(1 + f^2) \text{ for } t \in [0, b]; \]

then we can separate variables by dividing by \(1 + f^2\) and integrating the inequality from \(t = 0\) to \(t = t'\). The result is

\[ \tan^{-1}(f(t)) \leq t^2 + \tan^{-1}(f(0)). \]

However, once again we must question ourselves in doing what we really want to do: taking the \(\tan()\) of both sides. Well in general, if \(f \leq g\) and \(-\frac{\pi}{2} < f, g < \frac{\pi}{2}\) (or some other complete cycle), then \(\tan(f) \leq \tan(g)\). Unfortunately, this means that we need to impose a few more initial conditions: first we need to know the range of \(f\), second the value of \(f(0)\) needs to be stated, and lastly \(b\) needs to coincide with all these such that \(-\frac{\pi}{2} < t^2 + \tan^{-1}(f(0)) < \frac{\pi}{2}\) is satisfied.

**Example 2.** Again suppose we differentiate and obtain an ODI of the form

\[ f'(t) \leq (1 + 2f)te^{-t^2} \text{ for } t \in [a, b], f(t) \neq -\frac{1}{2}; \]

dividing and integrating this ODI yields

\[ \frac{1}{2} \log \left( \frac{1 + 2f}{1 + 2f(a)} \right) \leq \int_a^t ue^{-u^2} du = \frac{-1}{2} [e^{-t^2} - e^{-a^2}]. \]

Now after canceling the \(1/2's\) we want to take the exponential of both sides, and it seems obvious that we should be able to, but a little justification is in order. Let \(f \leq g\) on some interval. Then \(f - g\) is negative or zero, and so

\[ 0 < e^{f-g} \leq 1 \Rightarrow e^f \leq e^g. \]

Finally, after solving for \(f\) the the ODI becomes

\[ f(t) \leq \frac{1}{2}((1 + 2f(a))e^{-a^2-e^{-t^2}} - 1). \]

In Example 2, solving the ODI was a piece of cake because verifying that the exponential operator preserves inequalities was straightforward. The biggest problem with Example 1 is that the tangent function is not a linear operator like the differential operator. On the other hand, the \(\exp()\) operator is not linear either, but it still possesses similar qualities: it is a
group homomorphism (with respect to addition and multiplication of the real numbers). So I guess this process being fruitful depends largely upon what operator we have to deal with at the end.

This idea of a homomorphic function seems to work in general: if a function $\phi$ is a group isomorphism and $e_+ \leq \phi(f^{-1} \cdot g)$, then $\phi(f) \leq \phi(g)$, where $e_+$ is the identity element with respect to the binary operation in the domain of $\phi$. So for example, $e_+ = 0$ for $\phi = \log(x)$, and $e_+ = 1$ for $\phi = e^x$.

**Definition:** The inequality $f \leq g$ means $f(x) \leq g(x) \forall x \in [a, b]$.

**Proposition 2:** (Global Criteria)

Let $f, g \in X^1 = C^1([a, b]) := \{ f : [a, b] \to \mathbb{R}, f' \text{ continuous} \}$, then $f' \leq g'$ and $f(a) \leq g(a)$ if and only if $f \leq g$ and there exists a non-negative continuous function $\Phi : \mathbb{R} \to [0, +\infty)$ such that

$$(g - f)' \geq \Phi(g - f).$$

*Proof.* ($\Leftarrow$)

Suppose $f \leq g$ (so in particular $f(a) \leq g(a)$) and there exists a non-negative continuous function $\Phi$ such that $(g - f)' \geq \Phi(g - f) \geq 0$, then $(g - f)' = g' - f' \geq 0$.

($\Rightarrow$)

If

$$f' \leq g' \Rightarrow \int_a^x f' \leq \int_a^x g' \Rightarrow f(x) - f(a) \leq g(x) - g(a)$$

then $f(x) \leq g(x)$ since $f(a) \leq g(a)$. Also $f' \leq g'$ implies that $(g - f)' \geq 0$. So let $\Phi(g - f) = \min_{x \in [a, b]} \{(g(x) - f(x))'\}$. 

\[\Box\]

We can also construct some more complicated examples of $\Phi$. If $(g - f)'$ is concave on some interval, let $\Phi$ be an inscribing curve (or more simply just let $\Phi$ be the straight line connecting the endpoints on the interval). Combining the last idea, if $(g - f)'$ differentiable then $\Phi$ can alternate inscribing and circumscribing on concave and convex intervals (switching at each inflection point). Another thing to do if $(g - f)'$ is convex is to construct a polygonal curve in an Euler-method like fashion; start with the initial point and connect a tangent line to another point below the curve, and then connect another tangent line approximation to another point below the curve, and so on with evenly spaced points along the $x$-axis.
Let’s look at an explicit example. Define \( h = g - f \geq 0 \) and \( \Phi(x) = x^3 \); then from Proposition 2, if \( h \) satisfies the ODI \( h' \geq h^3 \) this implies that \( f' \leq g' \). From what was shown earlier we can separate variables (integrating from \( a \) to \( x \)) to end up with

\[
h \leq \frac{h(a)}{\sqrt{1 - 2h^2(a)(x - a)}}.
\]

This is the relationship \( f \) and \( g \) must satisfy.

As for general conditions on general operators, well, we really need to look at each operator individually. Even functions do not preserve inequalities such as \( \cos(x) \) and \( x^2 \) because they mask the sign, but the operator \( L[x] = -x \) does not work either. So for general abstract operators all we can do is make a good definition.

**Definition:** If \( L[u] \) is an operator \( L : X \rightarrow Y \) where \( X \) is a set of real-valued functions defined on \( I \subseteq \mathbb{R} \) and \( Y \) is also a set of real-valued functions on \( I \) and if for

\[
x \in I \ f(x) \leq g(x) \Rightarrow L[f] \leq L[g]
\]

on \( I \), then we say this inequality is preserved under the operation \( L \). If \( |L[f]| \leq |L[g]| \) is only true, then we say the magnitude of the inequality is preserved under the operation \( L \).

I hope that the interested reader will explore these and other means of seeing which operators preserve inequalities and how to solve more differential/integral inequalities.

**References**

