

Homology and dynamics in
quasi-isometric rigidity
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Gromov's program: Classify finitely generated groups according to their large scale geometric behavior.

Goal of these talks: Combine recent homological methods of Kevin Whyte with older dynamical methods of Benson Farb and myself, to obtain new quasi-isometric rigidity theorems:

Theorem (M-Whyte). *Given:*

- $S_g^1 =$ oriented surface of genus g minus a single point.
- $\mathcal{MCG}(S_g^1) =$ its mapping class group.
- $K =$ finitely generated group quasi-isometric to $\mathcal{MCG}(S_g^1)$

Then there exists a homomorphism

$$K \rightarrow \mathcal{MCG}(S_g^1)$$

with finite kernel and finite index image.

Theorem (Whyte). *If K is a finitely generated group quasi-isometric to $\mathbf{Z}^n \rtimes \mathrm{SL}(n, \mathbf{Z})$ then there is a homomorphism $K \rightarrow \mathbf{Z}^n \rtimes \mathrm{SL}(n, \mathbf{Z})$ with finite kernel and finite index image.*

Comments: Each of these theorems has a quantitatively more precise way to state the conclusion, with interesting corollaries.

Each of these theorems is about QI-rigidity for the “universal extension” of some group:

$\mathcal{MCG}(S_g^1) \approx \text{Aut}(\pi_1 S_g)$ is the universal extension of $\pi_1 S_g$.

$\mathbf{Z}^n \rtimes \text{GL}(n, \mathbf{Z})$ is the universal extension of \mathbf{Z}^n .

First theorem, about $\mathcal{MCG}(S_g^1)$, answers a special case of:

Conjecture 1. *If S is a surface of finite type (not exceptional) then for any f.g. group K quasi-isometric to $\mathcal{MCG}(S)$ there exists a homomorphism $K \rightarrow \mathcal{MCG}(S)$ with finite kernel and finite index image.*

Guess: the techniques used for the once-punctured case may not be too useful in the general case. But hey, you never know.

Also, the theorem for $\mathcal{MCG}(S_g^1) \approx \text{Aut}(\pi_1 S_g)$ leads one to ask about $\mathcal{MCG}(S_g^n)$ for $n \geq 2$, as well as for $\text{Aut}(F_n)$, each of which can be regarded as an extension with free kernel, as opposed to a $\pi_1 S_g$ kernel that arises for $\mathcal{MCG}(S_g^1)$. The difficulty is that the homological techniques we use do not apply: $\pi_1 S_g$ has a fundamental class, which F_n does not have.

Rough plan for these lectures:

Today Survey of results and techniques in quasi-isometric rigidity.

Tomorrow Whyte's techniques: uniformly finite homology applied to extension groups.

TDAT Surface group extensions.

TDATDAT Dynamical techniques: extensions of surface groups by pseudo-Anosov homeomorphisms.

If there is time, I will talk about the $\mathbf{Z}^n \rtimes \mathrm{SL}(n, \mathbf{Z})$ theorem.

Survey of results and techniques in QI-rigidity

A map $f: X \rightarrow Y$ of metric spaces is a *quasi-isometric embedding* if $\exists K \geq 1, C \geq 0$ such that

$$\frac{1}{K} \cdot d_X(x, y) - C \leq d_Y(fx, fy) \leq K \cdot d_X(x, y) + C$$

A *coarse inverse* for f is a quasi-isometry $\bar{f}: Y \rightarrow X$ s.t.

$$d_{\text{sup}}(\bar{f} \circ f, \text{Id}_X), \quad d_{\text{sup}}(f \circ \bar{f}, \text{Id}_Y) < \infty$$

A coarse inverse exists if and only if $\exists C' \geq 0$ such that $\forall y \in Y \exists x \in X$ such that

$$d_Y(fx, y) \leq C'$$

If this happens then $f: X \rightarrow Y$ is a *quasi-isometry*, and X, Y are *quasi-isometric* metric spaces.

Abbreviation: QI for quasi-isometric

Given a finitely generated group G , a *model space* for G is a metric space X on which G acts such that:

- X is *proper*: closed balls are compact.
- X is *geodesic*: any $x, y \in X$ are connected by a rectifiable path γ such that $\text{Length}(\gamma) = d(x, y)$.
- The action is properly discontinuous and cobounded.

Examples of model spaces:

- Cayley graph of G
- $X = \tilde{Y}$ where Y is compact, $\pi_1 Y \approx G$

Fact: If X, Y are two model spaces for G then X, Y are quasi-isometric. Also, any model space is QI to G with its word metric.

Consequence: A f.g. group G has a notion of geometry that is well-defined up to quasi-isometry, namely the geometry of any model space, or of G itself with a word metric.

Definition: Two f.g. groups are QI if, when equipped with their word metrics, they become QI as metric spaces.

Notation: Given

- \mathcal{G} = a collection of finitely generated groups, let

- $\langle \mathcal{G} \rangle$ = {all groups QI to some group in \mathcal{G} }.

Similarly, given

- \mathcal{X} = a collection of metric spaces, let

- $\langle \mathcal{X} \rangle$ = {all groups quasi-isometric to some metric space in \mathcal{X} }.

Examples of QI-rigidity theorems. General approach to Gromov's program:

- Given \mathcal{X} , describe $\langle \mathcal{X} \rangle$, preferably in simple algebraic or geometric terms that do not invoke the concept of quasi-isometry. Also, describe all of the quasi-isometry classes within $\langle \mathcal{X} \rangle$.
- In particular, identify interesting classes of groups \mathcal{G} that are *QI-rigid*, meaning

$$\mathcal{G} = \langle \mathcal{G} \rangle$$

There are many theorems describing interesting QI-rigid classes of groups, proved using an incredibly broad range of mathematical tools.

- Example: Gromov's polynomial growth theorem \implies

Theorem 2. *The class of virtually nilpotent groups is quasi-isometrically rigid. The class of virtually abelian groups is quasi-isometrically rigid, with one QI-class for each rank.*

Within the class of virtually nilpotent groups, there are many interesting QI-invariants:

- The Hirsch rank is a QI-invariant.
- The sequence of ranks of the abelian subquotients is a (finer) QI-invariant.
- There is an even finer QI-invariant of a virtually nilpotent Lie group G : Pansu proved that the asymptotic cone of G is a graded Lie algebra L , the so-called “associated graded Lie algebra” of G , and the isomorphism type of L is a quasi-isometry invariant that subsumes the sequence of subquotient ranks.
- This is still not the end of the story: recently Yehuda Shalom produced two finitely generated nilpotent groups which are not quasi-isometric but whose associated graded Lie algebras are isomorphic.

The full QI-classification of virtually nilpotent groups remains unknown.

- Example: Stallings' ends theorem \implies

Theorem 3. *The class of groups which splits over a finite group is quasi-isometrically rigid. For each $n \geq 2$, the class $\langle F_n \rangle$ consists of all groups that are virtually free of rank ≥ 2 .*

- Sullivan proved: any uniformly quasiconformal action on S^2 is quasiconformally conjugate to a conformal action. This implies:

Theorem 4 (Sullivan–Gromov). *$\langle \mathbf{H}^3 \rangle$ consists of all groups H for which there exists a homomorphism $H \rightarrow \text{Isom}(\mathbf{H}^3)$ with finite kernel and whose image is a co-compact lattice.*

This theorem is prototypical of a broad range of QI-rigidity theorems, including the theorems about $\text{MCG}(S_g^1)$ and $\mathbf{Z}^n \rtimes \text{GL}(n, \mathbf{Z})$.

However, contrast the conclusions: Sullivan–Gromov gives only a “topological” characterization of $\langle \mathbf{H}^3 \rangle$, which

does not serve to give us an effective list of those groups in $\langle \mathbf{H}^3 \rangle$. There is still no effective listing of the cocompact lattices acting on \mathbf{H}^3 .

- Example: Rich Schwarz' rigidity theorem for non-cocompact lattices in $\text{Isom}(\mathbf{H}^3)$.

To state the theorem we need some definitions.

The commensurator group. Given two groups G, H , a *commensuration* from G to H is an isomorphism from a finite index subgroup of G to a finite index subgroup of H .

Two commensurations are *equivalent* if they agree upon restriction to another finite index subgroup.

The *commensurator group* $\text{Comm}(G)$ is the set of self-commensurations of G up to equivalence, with the following group law: given commensurations

$$\phi: A \rightarrow B, \quad \psi: C \rightarrow D$$

restrict the range of ϕ and the domain of ψ to the finite index subgroup $B \cap C$, and then compose $\psi \circ \phi$.

The left action of G on itself by conjugation induces a homomorphism $G \rightarrow \text{Comm}(G)$.

Two groups G, H are *abstractly commensurable* if there exists a commensuration.

Theorem 5 (Schwarz). *Let G be a noncocompact, nonarithmetic lattice in $\text{Isom}(\mathbf{H}^3)$. Then $\langle G \rangle$ consists of those finitely generated groups H which are abstractly commensurable to G .*

More precisely, the homomorphism $G \rightarrow \text{Comm}(G)$ is an injection with finite index image, and $\langle G \rangle$ consists of those finitely generated groups H for which there exists a homomorphism $H \rightarrow \text{Comm}(G)$ with finite kernel and finite index image.

This theorem gives a very precise and effective enumeration of $\langle \mathcal{G} \rangle$, similar to the conclusion of our main theorems.

The general techniques of Sullivan-Gromov theorem, and of Schwarz' theorem give models for the proof of our main theorems.

Technique: the quasi-isometry group of a group.

Given

- $X =$ metric space, e.g. a model space for a finitely generated group. Let
- $\widehat{\text{QI}}(X) =$ set of self quasi-isometries of X , equipped with the operation of composition.
- Define an equivalence relation on $\widehat{\text{QI}}(X)$, where

$$f \sim g \quad \text{if} \quad d_{\text{sup}}(f, g) < \infty$$

- Composition descends to a group operation on the set of equivalence classes, giving a group

$\text{QI}(X) =$ the *quasi-isometry group* of X

Notation: $[f] =$ equivalence class of f in $\text{QI}(X)$.

Note: $[f]^{-1} = [\bar{f}]$.

For any quasi-isometry $f: X \rightarrow Y$ we obtain an isomorphism $\text{ad}_f: \text{QI}(X) \rightarrow \text{QI}(Y)$ defined by $\text{ad}_f[g] = [f \circ g \circ \bar{f}]$, where \bar{f} is any coarse inverse for f .

It follows that if G is a finitely generated group then the *quasi-isometry group* of G is well-defined up to isomorphism by taking it to be $\text{QI}(X)$ for any model space X of G .

The group $\text{QI}(G)$ is an important invariant of a group G , and it is often important to be able to compute it. In a little bit I'll give some examples of computations.

Facts about $\text{QI}(G)$:

- The left action of G on itself by multiplication,

$$L_g(h) = gh$$

induces a homomorphism $G \rightarrow \text{QI}(G)$ whose kernel is the *virtual center* of G , consisting of all elements $g \in G$ such that the centralizer of g has finite index in G .

- Every commensuration defines a natural quasi-isometry of G , well defined in $\text{QI}(G)$, thereby defining a homomorphism $\text{Comm}(G) \rightarrow \text{QI}(G)$.

- The left action of G on itself by conjugation,

$$C_g(h) = ghg^{-1}$$

defines a homomorphism $G \rightarrow \text{Comm}(G)$ whose kernel is also the virtual center.

- L_g and C_g are related by $d_{\text{sup}}(L_g, C_g) < \infty$; the constant is simply the word length of g .

- Therefore, the homomorphism $G \rightarrow \text{QI}(G)$ factors as

$$G \rightarrow \text{Comm}(G) \rightarrow \text{QI}(G)$$

Technique: quasi-actions Let

- $G =$ f.g. group
- $X =$ model space for G
- $H =$ f.g. group quasi-isometric to G .
- $\phi: H \rightarrow X$ a fixed quasi-isometry.
- Define $A: H \rightarrow \widehat{\text{QI}}(X)$ given by

$$A(h) = \phi \circ L_h \circ \bar{\phi}$$

This map has the following properties:

Quasi-action There exists constants $K \geq 1, C \geq 0$ such that

- The maps $A(h)$ are K, C quasi-isometries for all $h \in H$
- $d_{\text{sup}}(A(hh'), A(h) \circ A(h')) \leq C$ for all $h, h' \in H$
- $d_{\text{sup}}(A(\text{Id}), \text{Id}) \leq C$

and so we obtain a homomorphism $A: H \rightarrow \text{QI}(X)$.

Proper $\forall r \geq 0 \exists n$ such that if $B, B' \subset X$ have diameter $\leq r$ then

$$|\{h \in H \mid (A(h) \cdot B) \cap B' \neq \emptyset\}| \leq n$$

Cobounded $\exists s \geq 0$ such that $\forall x, y \in X \exists h \in H$ such that $d(A(h) \cdot x, y) \leq s$.

Given a group G and a model space X , a common strategy in investigating quasi-isometric rigidity of G is:

- Compute $\text{QI}(X)$.
- Describe those homomorphisms $H \rightarrow \text{QI}(X)$ arising from quasi-actions, called “uniform” homomorphism. If necessary, restrict to proper, cobounded quasi-actions. Try to “straighten” any such quasi-action.

Examples of QI-rigidity. Here are some examples of how this strategy is carried out, taken from the above examples:

Proof of the Sullivan–Gromov rigidity theorem. For group Γ in the quasi-isometry class of \mathbf{H}^3 , the boundary is $\partial\mathbf{H}^3 = S^2$.

First, one can calculate $\text{QI}(\mathbf{H}^3) = \text{QC}(S^2)$, the group of quasi-conformal homeomorphisms; this is classical result in quasiconformal geometry.

Second, the isometry group $\text{Isom}(\mathbf{H}^3) = \text{Conf}(S^2)$ is a uniform subgroup of $\text{QC}(S^2)$, and one proves that every uniform subgroup can be conjugated into $\text{Conf}(S^2)$. In other words, every quasi-action on \mathbf{H}^3 is quasiconjugate to an action. This is the heart of the proof.

The properties of “properness” and “coboundedness” are invariant under quasiconjugacy. It follows that if H

is a finitely generated group quasi-isometric to \mathbf{H}^3 then H has a proper, cobounded action on \mathbf{H}^3 .

In other words, there is a homomorphism $H \rightarrow \text{Isom}(\mathbf{H}^3)$ with finite kernel and discrete, cocompact image. \diamond

By contrast we now give:

Proof of the Schwartz rigidity theorem. Let G be a non-cocompact lattice in \mathbf{H}^3 .

The heart of the proof is essentially a calculation

$$\text{QI}(G) \approx \text{Comm}(G)$$

This calculation holds in both the arithmetic case and the nonarithmetic case, the difference being that the induced map $G \rightarrow \text{Comm}(G)$ has finite index image if and only if G is nonarithmetic. Assuming this to be the case, it follows that the homomorphism $\text{Comm}(G) \rightarrow \text{QI}(G)$ is an isomorphism and that the map $G \rightarrow \text{Comm}(G) \approx \text{QI}(G)$ is an injection with finite index image.

Schwartz' proof is actually a bit more quantitative, as follows.

G nonarithmetic $\implies \exists$ embedding

$$\text{Comm}(G) \hookrightarrow \text{Isom}(\mathbf{H}^3)$$

whose image is a noncocompact lattice Γ containing G with finite index, so that the injection $G \rightarrow \text{Comm}(G)$ agrees with the inclusion $G \hookrightarrow \Gamma$.

The hard bit of Schwarz proof is the following:

- $\forall K \geq 1, C \geq 0 \exists A \geq 0$ such that if $\phi: G \rightarrow G$ is a K, C quasi-isometry then there exists $\gamma \in \Gamma$ such that

$$d_{\text{sup}}(\phi, L_{\gamma}) \leq A$$

To be more precise, the sup distance on the left is a comparison of two different functions from G into Γ , one being $G \xrightarrow{\phi} G \hookrightarrow \Gamma$, and the other being $G \hookrightarrow \Gamma \xrightarrow{L_{\gamma}} \Gamma$.

Noting that any quasi-isometry of G extends to the finite index supergroup Γ , and that $\text{QI}(G) \approx \text{QI}(\Gamma)$, we can abstract this discussion as follows.

Consider a finitely generated group Γ , and suppose that the following holds:

Strong QI-rigidity: $\forall K \geq 1, C \geq 0 \exists A \geq 0$ such that if $\phi: \Gamma \rightarrow \Gamma$ is a K, C quasi-isometry then there exists $\gamma \in \Gamma$ such that

$$d_{\text{sup}}(\phi, L_{\gamma}) \leq A$$

This property, coupled with triviality of the virtual center (true for lattices in $\text{Isom}(\mathbf{H}^3)$ as well as for $\text{MCG}(S_g^1)$), immediately imply that the homomorphism $\Gamma \rightarrow \text{QI}(\Gamma)$ is an isomorphism.

To complete the proof of Schwarz' Theorem, we now apply the following fact:

Proposition 6. *If Γ is a strongly QI-rigid group whose virtual center is trivial, then for any finitely generated group H quasi-isometric to Γ there exists a homomorphism $H \rightarrow \Gamma$ with finite kernel and finite index image.*

Proof. As explained earlier, the left action of H on itself by translation can be quasiconjugated to a proper, cobounded quasi-action of H on Γ , which induces a homomorphism $\phi: H \rightarrow \text{QI}(\Gamma) = \Gamma$.

Let $K \geq 1$, $C \geq 0$ be uniform constants for the quasi-action of H on Γ .

Applying strong QI-rigidity of Γ , we obtain a constant A such that the (quasi-)action of each $h \in H$ on Γ is within sup distance A of left multiplication by $\phi(h)$. It immediately follows that the kernel of ϕ is finite, because the quasi-action of H is proper and so there are only finitely many elements $h \in H$ for which $\phi(h)$ is within distance A of the identity on Γ .

It also follows that the image of ϕ has finite index, because the quasi-action of H on Γ is cobounded, whereas the left action on Γ of any infinite index subgroup of Γ is not cobounded. \diamond

This completes the proof of Schwarz' Theorem. \diamond

This proof immediately yields an interesting corollary:

Corollary 7. *If Γ is strongly QI-rigid with trivial virtual center, then every commensuration of Γ is the restriction of an inner automorphism of Γ .*

Now I can give the more quantitative statement of the main theorem about $\mathcal{MCG}(S_g^1)$:

Theorem 8 (M-Whyte). *The group $\mathcal{MCG}(S_g^1)$ is strongly QI-rigid: for all $K \geq 1$, $C \geq 0$ there exists $A \geq 0$ such that for any K, C quasi-isometry $\phi: \mathcal{MCG}(S_g^1) \rightarrow \mathcal{MCG}(S_g^1)$ there exists $\gamma \in \mathcal{MCG}(S_g^1)$ for which $d_{\text{sup}}(\phi, L_\gamma) < A$.*

As an application, we get a new proof of a result of Ivanov:

Corollary 9 (Ivanov). *The injection $\mathcal{MCG}(S_g^1) \rightarrow \text{Comm}(S_g^1)$ is an isomorphism, that is, every commensuration of $\mathcal{MCG}(S_g^1)$ is the restriction of an inner automorphism.*