

Homology and dynamics in  
quasi-isometric rigidity  
Lecture Notes, Part I  
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**Gromov's program:** Classify finitely generated groups according to their large scale geometric behavior.

**Goal of these talks:** Combine recent homological methods of Kevin Whyte with older dynamical methods of Benson Farb and myself, to obtain new quasi-isometric rigidity theorems:

**Theorem (M-Whyte).** *Given:*

- $S_g^1 =$  oriented surface of genus  $g$  minus a single point.
- $\mathcal{MCG}(S_g^1) =$  its mapping class group.
- $K =$  finitely generated group quasi-isometric to  $\mathcal{MCG}(S_g^1)$

*Then there exists a homomorphism*

$$K \rightarrow \mathcal{MCG}(S_g^1)$$

*with finite kernel and finite index image.*

**Theorem (Whyte).** *If  $K$  is a finitely generated group quasi-isometric to  $\mathbf{Z}^n \rtimes \mathrm{SL}(n, \mathbf{Z})$  then there is a homomorphism  $K \rightarrow \mathbf{Z}^n \rtimes \mathrm{SL}(n, \mathbf{Z})$  with finite kernel and finite index image.*

**Comments:** Each of these theorems has a quantitatively more precise way to state the conclusion, with interesting corollaries.

Each of these theorems is about QI-rigidity for the “universal extension” of some group:

$\mathcal{MCG}(S_g^1) \approx \text{Aut}(\pi_1 S_g)$  is the universal extension of  $\pi_1 S_g$ .

$\mathbf{Z}^n \rtimes \text{GL}(n, \mathbf{Z})$  is the universal extension of  $\mathbf{Z}^n$ .

First theorem, about  $\mathcal{MCG}(S_g^1)$ , answers a special case of:

**Conjecture 1.** *If  $S$  is a surface of finite type (not exceptional) then for any f.g. group  $K$  quasi-isometric to  $\mathcal{MCG}(S)$  there exists a homomorphism  $K \rightarrow \mathcal{MCG}(S)$  with finite kernel and finite index image.*

Guess: the techniques used for the once-punctured case may not be too useful in the general case. But hey, you never know.

Also, the theorem for  $\mathcal{MCG}(S_g^1) \approx \text{Aut}(\pi_1 S_g)$  leads one to ask about  $\mathcal{MCG}(S_g^n)$  for  $n \geq 2$ , as well as for  $\text{Aut}(F_n)$ , each of which can be regarded as an extension with free kernel, as opposed to a  $\pi_1 S_g$  kernel that arises for  $\mathcal{MCG}(S_g^1)$ . The difficulty is that the homological techniques we use do not apply:  $\pi_1 S_g$  has a fundamental class, which  $F_n$  does not have.

## **Rough plan for these lectures:**

**Today** Survey of results and techniques in quasi-isometric rigidity.

**Tomorrow** Whyte's techniques: uniformly finite homology applied to extension groups.

**TDAT** Surface group extensions.

**TDATDAT** Dynamical techniques: extensions of surface groups by pseudo-Anosov homeomorphisms.

If there is time, I will talk about the  $\mathbf{Z}^n \rtimes \mathrm{SL}(n, \mathbf{Z})$  theorem.

## Survey of results and techniques in QI-rigidity

A map  $f: X \rightarrow Y$  of metric spaces is a *quasi-isometric embedding* if  $\exists K \geq 1, C \geq 0$  such that

$$\frac{1}{K} \cdot d_X(x, y) - C \leq d_Y(fx, fy) \leq K \cdot d_X(x, y) + C$$

A *coarse inverse* for  $f$  is a quasi-isometry  $\bar{f}: Y \rightarrow X$  s.t.

$$d_{\text{sup}}(\bar{f} \circ f, \text{Id}_X), \quad d_{\text{sup}}(f \circ \bar{f}, \text{Id}_Y) < \infty$$

A coarse inverse exists if and only if  $\exists C' \geq 0$  such that  $\forall y \in Y \exists x \in X$  such that

$$d_Y(fx, y) \leq C'$$

If this happens then  $f: X \rightarrow Y$  is a *quasi-isometry*, and  $X, Y$  are *quasi-isometric* metric spaces.

Abbreviation: QI for quasi-isometric

Given a finitely generated group  $G$ , a *model space* for  $G$  is a metric space  $X$  on which  $G$  acts such that:

- $X$  is *proper*: closed balls are compact.
- $X$  is *geodesic*: any  $x, y \in X$  are connected by a rectifiable path  $\gamma$  such that  $\text{Length}(\gamma) = d(x, y)$ .
- The action is properly discontinuous and cobounded.

Examples of model spaces:

- Cayley graph of  $G$
- $X = \tilde{Y}$  where  $Y$  is compact,  $\pi_1 Y \approx G$

Fact: If  $X, Y$  are two model spaces for  $G$  then  $X, Y$  are quasi-isometric. Also, any model space is QI to  $G$  with its word metric.

Consequence: A f.g. group  $G$  has a notion of geometry that is well-defined up to quasi-isometry, namely the geometry of any model space, or of  $G$  itself with a word metric.

Definition: Two f.g. groups are QI if, when equipped with their word metrics, they become QI as metric spaces.

Notation: Given

- $\mathcal{G}$  = a collection of finitely generated groups, let

- $\langle \mathcal{G} \rangle$  = {all groups QI to some group in  $\mathcal{G}$ }.

Similarly, given

- $\mathcal{X}$  = a collection of metric spaces, let

- $\langle \mathcal{X} \rangle$  = {all groups quasi-isometric to some metric space in  $\mathcal{X}$ }.

**Examples of QI-rigidity theorems.** General approach to Gromov's program:

- Given  $\mathcal{X}$ , describe  $\langle \mathcal{X} \rangle$ , preferably in simple algebraic or geometric terms that do not invoke the concept of quasi-isometry. Also, describe all of the quasi-isometry classes within  $\langle \mathcal{X} \rangle$ .
- In particular, identify interesting classes of groups  $\mathcal{G}$  that are *QI-rigid*, meaning

$$\mathcal{G} = \langle \mathcal{G} \rangle$$

There are many theorems describing interesting QI-rigid classes of groups, proved using an incredibly broad range of mathematical tools.

- Example: Gromov's polynomial growth theorem  $\implies$

**Theorem 2.** *The class of virtually nilpotent groups is quasi-isometrically rigid. The class of virtually abelian groups is quasi-isometrically rigid, with one QI-class for each rank.*

Within the class of virtually nilpotent groups, there are many interesting QI-invariants:

- The Hirsch rank is a QI-invariant.
- The sequence of ranks of the abelian subquotients is a (finer) QI-invariant.
- There is an even finer QI-invariant of a virtually nilpotent Lie group  $G$ : Pansu proved that the asymptotic cone of  $G$  is a graded Lie algebra  $L$ , the so-called “associated graded Lie algebra” of  $G$ , and the isomorphism type of  $L$  is a quasi-isometry invariant that subsumes the sequence of subquotient ranks.
- This is still not the end of the story: recently Yehuda Shalom produced two finitely generated nilpotent groups which are not quasi-isometric but whose associated graded Lie algebras are isomorphic.

The full QI-classification of virtually nilpotent groups remains unknown.

- Example: Stallings' ends theorem  $\implies$

**Theorem 3.** *The class of groups which splits over a finite group is quasi-isometrically rigid. For each  $n \geq 2$ , the class  $\langle F_n \rangle$  consists of all groups that are virtually free of rank  $\geq 2$ .*

- Sullivan proved: any uniformly quasiconformal action on  $S^2$  is quasiconformally conjugate to a conformal action. This implies:

**Theorem 4 (Sullivan–Gromov).**  *$\langle \mathbf{H}^3 \rangle$  consists of all groups  $H$  for which there exists a homomorphism  $H \rightarrow \text{Isom}(\mathbf{H}^3)$  with finite kernel and whose image is a co-compact lattice.*

This theorem is prototypical of a broad range of QI-rigidity theorems, including the theorems about  $\mathcal{MCG}(S_g^1)$  and  $\mathbf{Z}^n \rtimes \text{GL}(n, \mathbf{Z})$ .

However, contrast the conclusions: Sullivan–Gromov gives only a “topological” characterization of  $\langle \mathbf{H}^3 \rangle$ , which

does not serve to give us an effective list of those groups in  $\langle \mathbf{H}^3 \rangle$ . There is still no effective listing of the cocompact lattices acting on  $\mathbf{H}^3$ .

- Example: Rich Schwarz' rigidity theorem for non-cocompact lattices in  $\text{Isom}(\mathbf{H}^3)$ .

To state the theorem we need some definitions.

**The commensurator group.** Given two groups  $G, H$ , a *commensuration* from  $G$  to  $H$  is an isomorphism from a finite index subgroup of  $G$  to a finite index subgroup of  $H$ .

Two commensurations are *equivalent* if they agree upon restriction to another finite index subgroup.

The *commensurator group*  $\text{Comm}(G)$  is the set of self-commensurations of  $G$  up to equivalence, with the following group law: given commensurations

$$\phi: A \rightarrow B, \quad \psi: C \rightarrow D$$

restrict the range of  $\phi$  and the domain of  $\psi$  to the finite index subgroup  $B \cap C$ , and then compose  $\psi \circ \phi$ .

The left action of  $G$  on itself by conjugation induces a homomorphism  $G \rightarrow \text{Comm}(G)$ .

Two groups  $G, H$  are *abstractly commensurable* if there exists a commensuration.

**Theorem 5 (Schwarz).** *Let  $G$  be a noncocompact, nonarithmetic lattice in  $\text{Isom}(\mathbf{H}^3)$ . Then  $\langle G \rangle$  consists of those finitely generated groups  $H$  which are abstractly commensurable to  $G$ .*

*More precisely, the homomorphism  $G \rightarrow \text{Comm}(G)$  is an injection with finite index image, and  $\langle G \rangle$  consists of those finitely generated groups  $H$  for which there exists a homomorphism  $H \rightarrow \text{Comm}(G)$  with finite kernel and finite index image.*

This theorem gives a very precise and effective enumeration of  $\langle \mathcal{G} \rangle$ , similar to the conclusion of our main theorems.

The general techniques of Sullivan-Gromov theorem, and of Schwarz' theorem give models for the proof of our main theorems.

### **Technique: the quasi-isometry group of a group.**

Given

- $X =$  metric space, e.g. a model space for a finitely generated group. Let
- $\widehat{\text{QI}}(X) =$  set of self quasi-isometries of  $X$ , equipped with the operation of composition.
- Define an equivalence relation on  $\widehat{\text{QI}}(X)$ , where

$$f \sim g \quad \text{if} \quad d_{\text{sup}}(f, g) < \infty$$

- Composition descends to a group operation on the set of equivalence classes, giving a group

$\text{QI}(X) =$  the *quasi-isometry group* of  $X$

Notation:  $[f] =$  equivalence class of  $f$  in  $\text{QI}(X)$ .

Note:  $[f]^{-1} = [\bar{f}]$ .

For any quasi-isometry  $f: X \rightarrow Y$  we obtain an isomorphism  $\text{ad}_f: \text{QI}(X) \rightarrow \text{QI}(Y)$  defined by  $\text{ad}_f[g] = [f \circ g \circ \bar{f}]$ , where  $\bar{f}$  is any coarse inverse for  $f$ .

It follows that if  $G$  is a finitely generated group then the *quasi-isometry group* of  $G$  is well-defined up to isomorphism by taking it to be  $\text{QI}(X)$  for any model space  $X$  of  $G$ .

The group  $\text{QI}(G)$  is an important invariant of a group  $G$ , and it is often important to be able to compute it. In a little bit I'll give some examples of computations.

Facts about  $\text{QI}(G)$ :

- The left action of  $G$  on itself by multiplication,

$$L_g(h) = gh$$

induces a homomorphism  $G \rightarrow \text{QI}(G)$  whose kernel is the *virtual center* of  $G$ , consisting of all elements  $g \in G$  such that the centralizer of  $g$  has finite index in  $G$ .

- Every commensuration defines a natural quasi-isometry of  $G$ , well defined in  $\text{QI}(G)$ , thereby defining a homomorphism  $\text{Comm}(G) \rightarrow \text{QI}(G)$ .

- The left action of  $G$  on itself by conjugation,

$$C_g(h) = ghg^{-1}$$

defines a homomorphism  $G \rightarrow \text{Comm}(G)$  whose kernel is also the virtual center.

- $L_g$  and  $C_g$  are related by  $d_{\text{sup}}(L_g, C_g) < \infty$ ; the constant is simply the word length of  $g$ .

- Therefore, the homomorphism  $G \rightarrow \text{QI}(G)$  factors as

$$G \rightarrow \text{Comm}(G) \rightarrow \text{QI}(G)$$

**Technique: quasi-actions** Let

- $G =$  f.g. group
- $X =$  model space for  $G$
- $H =$  f.g. group quasi-isometric to  $G$ .
- $\phi: H \rightarrow X$  a fixed quasi-isometry.
- Define  $A: H \rightarrow \widehat{\text{QI}}(X)$  given by

$$A(h) = \phi \circ L_h \circ \bar{\phi}$$

This map has the following properties:

**Quasi-action** There exists constants  $K \geq 1, C \geq 0$  such that

- The maps  $A(h)$  are  $K, C$  quasi-isometries for all  $h \in H$
- $d_{\text{sup}}(A(hh'), A(h) \circ A(h')) \leq C$  for all  $h, h' \in H$
- $d_{\text{sup}}(A(\text{Id}), \text{Id}) \leq C$

and so we obtain a homomorphism  $A: H \rightarrow \text{QI}(X)$ .

**Proper**  $\forall r \geq 0 \exists n$  such that if  $B, B' \subset X$  have diameter  $\leq r$  then

$$|\{h \in H \mid (A(h) \cdot B) \cap B' \neq \emptyset\}| \leq n$$

**Cobounded**  $\exists s \geq 0$  such that  $\forall x, y \in X \exists h \in H$  such that  $d(A(h) \cdot x, y) \leq s$ .

Given a group  $G$  and a model space  $X$ , a common strategy in investigating quasi-isometric rigidity of  $G$  is:

- Compute  $\text{QI}(X)$ .
- Describe those homomorphisms  $H \rightarrow \text{QI}(X)$  arising from quasi-actions, called “uniform” homomorphism. If necessary, restrict to proper, cobounded quasi-actions. Try to “straighten” any such quasi-action.

**Examples of QI-rigidity.** Here are some examples of how this strategy is carried out, taken from the above examples:

*Proof of the Sullivan–Gromov rigidity theorem.* For group  $\Gamma$  in the quasi-isometry class of  $\mathbf{H}^3$ , the boundary is  $\partial\mathbf{H}^3 = S^2$ .

First, one can calculate  $\text{QI}(\mathbf{H}^3) = \text{QC}(S^2)$ , the group of quasi-conformal homeomorphisms; this is classical result in quasiconformal geometry.

Second, the isometry group  $\text{Isom}(\mathbf{H}^3) = \text{Conf}(S^2)$  is a uniform subgroup of  $\text{QC}(S^2)$ , and one proves that every uniform subgroup can be conjugated into  $\text{Conf}(S^2)$ . In other words, every quasi-action on  $\mathbf{H}^3$  is quasiconjugate to an action. This is the heart of the proof.

The properties of “properness” and “coboundedness” are invariant under quasiconjugacy. It follows that if  $H$

is a finitely generated group quasi-isometric to  $\mathbf{H}^3$  then  $H$  has a proper, cobounded action on  $\mathbf{H}^3$ .

In other words, there is a homomorphism  $H \rightarrow \text{Isom}(\mathbf{H}^3)$  with finite kernel and discrete, cocompact image.  $\diamond$

By contrast we now give:

*Proof of the Schwartz rigidity theorem.* Let  $G$  be a non-cocompact lattice in  $\mathbf{H}^3$ .

The heart of the proof is essentially a calculation

$$\text{QI}(G) \approx \text{Comm}(G)$$

This calculation holds in both the arithmetic case and the nonarithmetic case, the difference being that the induced map  $G \rightarrow \text{Comm}(G)$  has finite index image if and only if  $G$  is nonarithmetic. Assuming this to be the case, it follows that the homomorphism  $\text{Comm}(G) \rightarrow \text{QI}(G)$  is an isomorphism and that the map  $G \rightarrow \text{Comm}(G) \approx \text{QI}(G)$  is an injection with finite index image.

Schwartz' proof is actually a bit more quantitative, as follows.

$G$  nonarithmetic  $\implies \exists$  embedding

$$\text{Comm}(G) \hookrightarrow \text{Isom}(\mathbf{H}^3)$$

whose image is a noncocompact lattice  $\Gamma$  containing  $G$  with finite index, so that the injection  $G \rightarrow \text{Comm}(G)$  agrees with the inclusion  $G \hookrightarrow \Gamma$ .

The hard bit of Schwarz proof is the following:

- $\forall K \geq 1, C \geq 0 \exists A \geq 0$  such that if  $\phi: G \rightarrow G$  is a  $K, C$  quasi-isometry then there exists  $\gamma \in \Gamma$  such that

$$d_{\text{sup}}(\phi, L_\gamma) \leq A$$

To be more precise, the sup distance on the left is a comparison of two different functions from  $G$  into  $\Gamma$ , one being  $G \xrightarrow{\phi} G \hookrightarrow \Gamma$ , and the other being  $G \hookrightarrow \Gamma \xrightarrow{L_\gamma} \Gamma$ .

Noting that any quasi-isometry of  $G$  extends to the finite index supergroup  $\Gamma$ , and that  $\text{QI}(G) \approx \text{QI}(\Gamma)$ , we can abstract this discussion as follows.

Consider a finitely generated group  $\Gamma$ , and suppose that the following holds:

**Strong QI-rigidity:**  $\forall K \geq 1, C \geq 0 \exists A \geq 0$  such that if  $\phi: \Gamma \rightarrow \Gamma$  is a  $K, C$  quasi-isometry then there exists  $\gamma \in \Gamma$  such that

$$d_{\text{sup}}(\phi, L_\gamma) \leq A$$

This property, coupled with triviality of the virtual center (true for lattices in  $\text{Isom}(\mathbf{H}^3)$  as well as for  $\mathcal{MCG}(S_g^1)$ ), immediately imply that the homomorphism  $\Gamma \rightarrow \text{QI}(\Gamma)$  is an isomorphism.

To complete the proof of Schwarz' Theorem, we now apply the following fact:

**Proposition 6.** *If  $\Gamma$  is a strongly QI-rigid group whose virtual center is trivial, then for any finitely generated group  $H$  quasi-isometric to  $\Gamma$  there exists a homomorphism  $H \rightarrow \Gamma$  with finite kernel and finite index image.*

*Proof.* As explained earlier, the left action of  $H$  on itself by translation can be quasiconjugated to a proper, cobounded quasi-action of  $H$  on  $\Gamma$ , which induces a homomorphism  $\phi: H \rightarrow \text{QI}(\Gamma) = \Gamma$ .

Let  $K \geq 1$ ,  $C \geq 0$  be uniform constants for the quasi-action of  $H$  on  $\Gamma$ .

Applying strong QI-rigidity of  $\Gamma$ , we obtain a constant  $A$  such that the (quasi-)action of each  $h \in H$  on  $\Gamma$  is within sup distance  $A$  of left multiplication by  $\phi(h)$ . It immediately follows that the kernel of  $\phi$  is finite, because the quasi-action of  $H$  is proper and so there are only finitely many elements  $h \in H$  for which  $\phi(h)$  is within distance  $A$  of the identity on  $\Gamma$ .

It also follows that the image of  $\phi$  has finite index, because the quasi-action of  $H$  on  $\Gamma$  is cobounded, whereas the left action on  $\Gamma$  of any infinite index subgroup of  $\Gamma$  is not cobounded.  $\diamond$

This completes the proof of Schwarz' Theorem.  $\diamond$

This proof immediately yields an interesting corollary:

**Corollary 7.** *If  $\Gamma$  is strongly QI-rigid with trivial virtual center, then every commensuration of  $\Gamma$  is the restriction of an inner automorphism of  $\Gamma$ .*

Now I can give the more quantitative statement of the main theorem about  $\mathcal{MCG}(S_g^1)$ :

**Theorem 8 (M-Whyte).** *The group  $\mathcal{MCG}(S_g^1)$  is strongly QI-rigid: for all  $K \geq 1$ ,  $C \geq 0$  there exists  $A \geq 0$  such that for any  $K, C$  quasi-isometry  $\phi: \mathcal{MCG}(S_g^1) \rightarrow \mathcal{MCG}(S_g^1)$  there exists  $\gamma \in \mathcal{MCG}(S_g^1)$  for which  $d_{\text{sup}}(\phi, L_\gamma) < A$ .*

As an application, we get a new proof of a result of Ivanov:

**Corollary 9 (Ivanov).** *The injection  $\mathcal{MCG}(S_g^1) \rightarrow \text{Comm}(S_g^1)$  is an isomorphism, that is, every commensuration of  $\mathcal{MCG}(S_g^1)$  is the restriction of an inner automorphism.*