

Homology and dynamics in
quasi-isometric rigidity
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Fiber preserving quasi-isometries

Benson Farb and I proved QI-rigidity for fundamental groups of graphs of groups, where each vertex and edge group is (virtually) π_1 of an aspherical n -manifold for fixed integer $n \geq 0$.

Example: $F_2 \times \mathbf{Z}^n$.

Start of proof: Pick a nice model space for $F_2 \times \mathbf{Z}^n$, namely, $T \times \mathbf{R}^n$ where $T =$ Cayley tree of F_2 .

Study quasi-actions on $T \times \mathbf{R}^n$.

Start by proving that each quasi-isometry of $T \times \mathbf{R}^n$ coarsely preserves the \mathbf{R}^n fibers:

Theorem 1 (Farb-M). $\forall K, C \exists A$ such that if $\phi: T \times \mathbf{R}^n \rightarrow T \times \mathbf{R}^n$ is a K, C quasi-isometry, then $\forall t \in T \exists t' \in T$ such that $d_{\text{Haus}}(\phi(t \times \mathbf{R}^n), t' \times \mathbf{R}^n) < A$.

More generally, true replacing the product $T \times \mathbf{R}^n$ by a fiber bundle over T whose fiber is a uniformly contractible manifold.

Today: Kevin Whyte's proof. His proof to situations where the base space T of the fibration is replaced by certain higher dimensional complexes, for example:

- thick building
- model space for $\mathcal{MCG}(S_g)$, over which a model space for $\mathcal{MCG}(S_g^1)$ fibers, with fiber \mathbf{H}^2 .
- model space for $\mathrm{SL}(n, \mathbf{Z})$, over which a model space for $\mathbf{Z}^n \rtimes \mathrm{GL}(n, \mathbf{Z})$ fibers, with fiber \mathbf{R}^n .

Idea of the proof:

- Subset of the form $(\text{line in } T) \times \mathbf{R}^n \approx \mathbf{R}^{n+1}$ is the support of a “top dimensional uniformly finite homology class”.
- A quasi-isometry acts on such classes, coarsely preserving their supports.
- Each fiber is the intersection of some finite number of these supports, and so the fibers are preserved.

By the way, in this proof it is necessary that the fiber be a uniformly contractible *manifold*, on which there is a “uniformly finite” fundamental class of full support. (For those who live in outer space, that’s why the proof does not apply to the extension $1 \rightarrow F_n \rightarrow \text{Aut}(F_n) \rightarrow \text{Out}(F_n) \rightarrow 1$, but the fiber F_n is not a manifold, and there is no uniformly finite fundamental class of full support).

In order to make this proof rigorous, we have to discuss:

1. Uniformly finite homology
2. Top dimensional supports
3. Application to fiber bundles

1. Uniformly finite homology

$X =$ simplicial complex. Fix a geodesic metric in which each simplex is a regular Euclidean simplex with side length 1.

X is *uniformly locally finite* or ULF if $\exists A \geq 0$ s.t. the link of each simplex contains at most A simplices.

X is *uniformly contractible* or UC if $\forall r > 0 \exists s(r) \geq 0$ s.t. each subset $A \subset X$ with $\text{diam}(A) \leq r$ is contractible to a point inside $N_s(A)$. The function $s(r)$ is called a *gauge* of uniform contractibility.

- Example: any tree is uniformly contractible.
- Example: $X =$ contractible simplicial complex. If there is a cocompact, simplicial group action on X (e.g. $X =$ universal cover of a compact, aspherical simplicial complex), then X is UC and ULF.

Simplicial uniformly finite homology. If X is ULF, define $H_n^{\text{suf}}(X)$: a chain in $C_n^{\text{suf}}(X)$ is a uniformly bounded assignments of integers to n -simplices. Since X is ULF, $\partial C_n^{\text{suf}}(X) \rightarrow C_{n-1}^{\text{suf}}(X)$ is defined, and clearly $\partial\partial = 0$.

Theorem 2. $H_*^{\text{suf}}(X)$ is a quasi-isometry invariant among UC, ULF simplicial complexes.

Step 1: Uniformly finite homology. Define the n th Rips complex $R^n(X)$: one simplex for each ordered $k+1$ -tuple of vertices with diameter $\leq n$.

Note: $R^1(X) = X$.

Since X is ULF, follows that $R^n(X)$ is ULF. The inclusions

$$X = R^1(X) \subset R^2(X) \subset \dots$$

therefore induce homomorphisms

$$H_*^{\text{suf}}(R^1(X)) \rightarrow H_*^{\text{suf}}(R^2(X)) \rightarrow H_*^{\text{suf}}(R^3(X)) \dots$$

Define the uniformly finite homology

$$H_*^{\text{uf}}(X) = \lim_{k \rightarrow \infty} H_*^{\text{suf}}(R^k(X))$$

In terms of chains: we have a direct system

$$C_*^{\text{suf}}(R^1(X)) \rightarrow C_*^{\text{suf}}(R^2(X)) \rightarrow C_*^{\text{suf}}(R^3(X)) \rightarrow \dots$$

so we can take the direct limit

$$C_*^{\text{uf}}(X) = \lim_{k \rightarrow \infty} C_*^{\text{suf}}(R^k(X))$$

The boundary homomorphism is defined, and the homology of this chain complex is canonically isomorphic to $H_*^{\text{uf}}(X)$.

Step 2: $H_*^{\text{uf}}(X) = H_*^{\text{suf}}(X)$. The identity $i: X \rightarrow X = R^1(X)$ induces a chain map $i: C_*^{\text{suf}}(X) \rightarrow C_*^{\text{uf}}(X)$.

Using that X is UC, we'll define a chain map

$$j: C_*^{\text{uf}}(X) \rightarrow C_*^{\text{suf}}(X)$$

which is a uniform chain homotopy inverse to the inclusion $C_*^{\text{suf}}(X) \rightarrow C_*^{\text{uf}}(X)$.

Idea for definition of j : Connect-the-dots

Consider:

- a 1-simplex in $C_1^{\text{uf}}(X)$, which means
- a 1-simplex σ in $R^k(X)$ for some k , which means
- $\sigma = (u, v)$ with $d(u, v) \leq k$.

Connecting the dots, we get a 1-chain $j(\sigma)$ in X with boundary $v - u$, consisting of at most k 1-simplices.

More generally, given a simplicial uniformly finite 1-chain $\sum a_\sigma \sigma$ in $R^k(X)$, the infinite sum

$$j\left(\sum a_\sigma \sigma\right) = \sum a_\sigma j(\sigma)$$

is defined because it is locally finite: for each simplex τ in X there are finitely many terms of the sum $\sum a_\sigma j(\sigma)$ which assign a nonzero coefficient to τ .

Consider:

- a 2-simplex σ in $C_2^{\text{uf}}(X)$, which means
- $\sigma = (u, v, w)$, where u, v, w have pairwise distances at most k .

Now connect the 2-dimensional dots: the 1-chain

$$j(u, v) + j(v, w) + j(w, u)$$

is a cycle.

Its support is a subset of diameter at most $3k/2$, and so

$$j(u, v) + j(v, w) + j(w, u) = \partial j(\sigma)$$

for some 2-chain $j(\sigma)$ supported on a subset of diameter at most $s(3k/2)$, where s is a gauge of uniform contractibility.

More generally, the boundary of a simplicially uniformly finite 2-chain in $R^2(X)$, is again defined as a locally finite infinite sum.

Now continue the definition of j by induction. Similarly, using connect the dots and induction, construct chain homotopies between identity and ji , and similarly for ij .

Step 3: Consider a quasi-isometry $\phi: X \rightarrow Y$ with coarse inverse $\bar{\phi}: Y \rightarrow X$, both K, C quasi-isometries, and C -coarse inverses of each other. Moving a bounded distance, may assume $\phi, \bar{\phi}$ take vertices to vertices.

If $d(u, v) = 1$ then $d(\phi u, \phi v) \leq p = K + C$.

Get an induced simplicial map $X = R^1(X) \rightarrow R^p(X)$, inducing a chain map

$$\phi_{\#}: C_*^{\text{suf}}(R^1(X)) \rightarrow C_*^{\text{suf}}(R^p(X))$$

In the backwards direction:

if $d(u', v') \leq p$ in Y , then $d(\bar{\phi}u', \bar{\phi}v') \leq p' = Kp + C$.

So, we get an induced simplicial map

$$R^p(Y) \rightarrow R^{p'}(X)$$

inducing a chain map

$$\bar{\phi}_{\#} : C_*^{\text{suf}}(R^p(Y)) \rightarrow C_*^{\text{suf}}(R^{p'}(X))$$

The composition $\bar{\phi} \circ \phi$ is C -close to the identity map on vertices of X , inducing a chain map

$$\bar{\phi}_{\#} \circ \bar{\phi} : C_*^{\text{suf}}(R^1(X)) \rightarrow C_*^{\text{suf}}(R^{p'}(X))$$

Use connect-the-dots to show that the induced chain map is chain homotopic to the inclusion map.

A similar argument applies to the composition $\phi \circ \bar{\phi}$.

This finishes the proof that H_*^{suf} is a QI invariant.

2. Top dimensional supports. Suppose now:

- $X = \text{UC}$, ULF simplicial complex of dimension d .
- So, each class $c \in H_d^{\text{uf}}(X)$ is represented by a unique d -cycle in $C_d^{\text{supf}}(X)$, also denoted c . Its support $\text{supp}(c)$ is therefore a well-defined subset of X .

Proposition 3. *With X as above, every QI of X coarsely respects supports of classes in $H_d^{\text{uf}}(X)$.*

More precisely: $\forall K, C \exists A$ such that if $\phi: X \rightarrow X$ is a K, C quasi-isometry, and if $c \in H_d^{\text{uf}}(X)$, then

$$d_{\text{Haus}}(\phi(\text{supp}(c)), \text{supp}(\phi_*(c))) < A$$

Most QI-rigidity theorems have a similar step: find some collection \mathcal{C} of objects in the model space which are coarsely respected by quasi-isometries:

- $\forall K, C \exists A$ s.t. for each K, C quasi-isometry $\phi: X \rightarrow X$, and for each object $c \in \mathcal{C}$ there exists an object $c' \in \mathcal{C}$ such that

$$d_{\text{Haus}}(\phi(c), c') < A$$

Proof. Use same notation c for the unique cycle in $C_d^{\text{suf}}(X)$ representing c .

Changing ϕ by a bounded distance, to takes vertices to vertices.

Get an induced chain map

$$\phi_{\#} : C^{\text{suf}}(X) \rightarrow C^{\text{suf}}(R^p(X))$$

Note: $\forall c \in C^{\text{suf}}(X)$, $\text{supp}(\phi_{\#}(c))$ contained in a uniformly bounded neighborhood of $\phi(\text{supp}(c))$.

In other words:

- $\phi_{\#}$ induces coarse inclusion of supports.

Compose with the connect-the-dots map

$$C^{\text{suf}}(R^p(X)) \rightarrow C^{\text{suf}}(X)$$

which also induces coarse inclusion of supports.

Obtain an induced map

$$\phi_{\#\#\#}: C^{\text{suf}}(X) \rightarrow C^{\text{suf}}(X)$$

which also induces coarse inclusion of supports.

Since top dimensional supports are unique, it follows that

$$\phi(\text{supp}(c)) \subset N_A(\text{supp}(\phi_{\#\#\#}(c)))$$

for some uniform constant A .

To get the inverse inclusion, applying the same argument to a coarse inverse $\bar{\phi}$ we have

$$\bar{\phi}(\text{supp}(\phi_*(c))) \subset N_A(\text{supp}(\bar{\phi}_*\phi_*(c))) = N_A(\text{supp}(c))$$

where the last equation follows from uniqueness of supports.

Now apply ϕ to both sides of this equation:

$$\begin{aligned} \text{supp}(\phi_*(c)) &\subset N_{A'}(\phi\bar{\phi}(\text{supp}(\phi_*(c)))) \\ &\subset N'_{A'}(\phi(N_A(\text{supp}(c)))) \\ &\subset N_{A''}(\phi(\text{supp}(c))) \end{aligned}$$



3. Application to certain fiber bundles. Consider:

- fiber bundle $\pi: E \rightarrow B$ with fiber F_x over each $x \in B$.

We assume:

- E, B are UC, ULF simplicial complexes and π is a simplicial map.
- Each fiber F_x is a manifold, of dimension n .
- For each vertex x , the subcomplex F_x is UC, with gauge independent of x .

Follows that:

- For each k -simplex σ , the $k+n$ simplices of $\pi^{-1}(\sigma)$ that are not contained in $\pi^{-1}(\partial\sigma)$, intersected with the fiber $F_\sigma = \pi^{-1}(\text{barycenter}(\sigma))$, define a cellular structure on F_σ which is UC, with gauge independent of σ .

Let $d = \dim(B)$, $n = \dim(F)$, $d + n = \dim(E)$.

Make the following assumption about the top dimensional, uniformly finite homology $H_d^{\text{uf}}(B)$:

Top dimensional classes in B separate points:

$\exists r > 0$ so that $\forall s > 0 \exists D > 0$ so that:

for any $x, y \in B$ with $d(x, y) > D$, there is a top dimensional class $c \in H_d^{\text{uf}}(B)$ such that

$$d(\text{supp}(c), x) \leq r \quad \text{and} \quad d(\text{supp}(c), y) > s$$

Example: $T \times \mathbf{R}^n$, where $T =$ Cayley tree of F_2 . In the base space T , each bi-infinite line is the support of a top dimensional class, and lines in T clearly separate points.

Example next time: model space for $\mathcal{MCG}(S_g^1)$, fibering over model space for $\mathcal{MCG}(S_g)$, with fiber \mathbf{H}^2 .

Main result for today: every quasi-isometry of E coarsely preserves fibers:

Theorem 4 (Whyte). *Consider the fibration $F \rightarrow E \rightarrow B$ as above, and assume that top dimensional classes in $H_d^{\text{uf}}(B)$ coarsely separate points. For all K, C there exists A such that if $\phi: E \rightarrow E$ is a K, C quasi-isometry, then for each $x \in B$ there exists $x' \in B'$ such that*

$$d_{\text{Haus}}(\phi(F_x), F_{x'}) \leq A$$

Proof. Key observation: the support of every top- d class in E is saturated by fibers.

To be precise: for every top- d class $c \in H_{d+n}^{\text{uf}}(E)$, there exists a unique top- d class $c' = \pi(c) \in H_d^{\text{uf}}(B)$, such that

$$\text{supp}(c) = \pi^{-1}(\text{supp}(c'))$$

Why? For each d -simplex $\sigma \subset B$, $\pi^{-1}(\sigma)$ is a manifold with boundary of dimension $d + n$.

So, for any class c of dimension $d+n$, if $\text{supp}(c)$ contains some $d+n$ simplex in $\pi^{-1}(\sigma)$, it follows that $\text{supp}(c)$ contains all of $\pi^{-1}(\sigma)$.

Converse is also true: for each top- d class $c' \in H_d^{\text{uf}}(B)$ there exists a top- d class in E , denote $c = \pi^{-1}(c') \in H_{d+n}^{\text{uf}}(E)$ such that $\text{supp}(c) = \pi^{-1}(\text{supp}(c'))$: over each simplex $\sigma \subset \text{supp}(c')$, weight all the simplices in $\pi^{-1}(\sigma)$ with the same weight as σ , using a coherent orientation of fibers to choose the sign.

Thus, the projection π induces an isomorphism

$$H_{d+n}^{\text{uf}}(E) \rightarrow H_d^{\text{uf}}(B)$$

so that a $d+n$ -cycle c in E , and the corresponding d -cycle c' in B , are related by

$$\text{supp}(c) = \pi^{-1}(\text{supp}(c'))$$

Now we use the property that supports of top dimensional classes in B separate points. Up to changing constants, it follows that:

Supports of top-d classes in E separate fibers: there exists $r > 0$ such that for all $s > 0$ there exists $D > 0$ such that given fibers F_x, F_y with $d_{\text{Haus}}(F_x, F_y) > D$, there is a top dimensional class $c \in H_{d+n}^{\text{uf}}(E)$ so that $F_x \subset N_r(\text{supp}(c))$ but $F_y \cap N_r(\text{supp}(c)) = \emptyset$.

From this it follows that fibers in E are coarsely respected by a quasi-isometry. Here are the details.

Fix a K, C quasi-isometry ϕ and a fiber $F_x, x \in G$. We want to show that $\phi(F_x)$ is uniformly Hausdorff close to some fiber $F_{x'}$.

Fix some $R > r$, to be chosen later, and let \mathcal{C}_x denote the collection of classes $c \in H_{d+n}^{\text{uf}}(E)$ such that $F_x \subset N_R(\text{supp}(c))$. From the fact that top dimensional classes in E separate fibers, it follows that F_x has (uniformly) finite Hausdorff distance from the set of points $\xi \in E$ such that $\xi \in N_R(\text{supp}(c))$ for all $c \in \mathcal{C}_x$.

Notation: let $\hat{\mathcal{C}}_x = \{\phi_{\#}(c) \mid c \in \mathcal{C}_x\}$, and let $\hat{c} = \phi_{\#}(c)$.

By applying Proposition 3, the Hausdorff distance between $\phi(\text{supp}(c))$ and $\text{supp}(\hat{c})$ is at most a constant A , for any $c \in \mathcal{C}_x$. Thus for any $\hat{c} \in \hat{\mathcal{C}}_x$ we have

$$\phi(F_x) \subset N_{R'}(\text{supp}(\hat{c}))$$

where $R' = KR + C + A$. Now we say how large to choose R , namely, so that $R' > r$.

It now follows that $\phi(F_x)$ has (uniformly) finite Hausdorff distance from the set \mathcal{F} of points $\eta \in E$ such that $\eta \in N_{R'}(\text{supp}(\hat{c}))$ for all $\hat{c} \in \hat{\mathcal{C}}_x$. But the set \mathcal{F} is clearly a union of fibers of E . Pick one fiber $F_{x'}$ in \mathcal{F} . Taking $s = R'$ in the definition of coarse separation of fibers, there is a resulting D . If $F_{y'}$ is a fiber whose distance from $F_{x'}$ is more than D , it follows that $F_{y'}$ is not contained in \mathcal{F} , because that would violate coarse separation of fibers. This shows that the set \mathcal{F} contains $F_{x'}$ and is contained in the D -neighborhood of $F_{x'}$, that is, \mathcal{F} has Hausdorff distance at most D from $F_{x'}$. But \mathcal{F} also has (uniformly) finite Hausdorff distance from $\phi(F_x)$, and so $\phi(F_x)$ has (uniformly) finite Hausdorff distance from $F_{x'}$.

This finishes the proof of Whyte's theorem. ◇