Homology and dynamics in quasi-isometric rigidity Lecture Notes, Part III Durham, July 7, 2003

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Last time: Proving that quasi-isometries coarsely preserve fibers

Given:

- fiber bundle $F \to E \to B$, with fiber F_x over each $x \in B$.
- $d = \dim(B)$, $n = \dim(F)$, $d + n = \dim(E)$.
- E, B are UC, ULF simplicial complexes
- π is a simplicial map.
- Each fiber F_x is a manifold, of dimension n.
- For each vertex x, the subcomplex F_x is UC, with gauge independent of x.
- Top dimensional, uniformly finite homology classes in B coarsely separate points.

Here is a definition of the last property (the definition I gave in the lecture for Part II was a tad too strong, and is now replaced by a more appropriate, weaker form of the definition):

Top-d classes in $H_d^{\mathsf{uf}}(B)$ coarsely separate points:
$\exists r > 0$ so that $\forall s > 0$ $\exists D > 0$ so that:
for any $x, y \in B$ with $d(x, y) > D$, there is a top dimensional class $c \in H_d^{\mathsf{uf}}(B)$ such that

$$d(\mathsf{supp}(c), x) \leq r \quad \text{and} \quad d(\mathsf{supp}(c), y) > s$$

With this definition, what was proved last time was:

Theorem 1 (Whyte). *If top dimensional classes in B coarsely separate points, then quasi-isometries of E coarsely preserve fibers.*

Mapping class groups

Today: I'll explain Whyte's method for using Mess subgroups of mapping class groups to verify that quasi-isometries of $\mathcal{MCG}(S_g^1)$ do, indeed, coarsely preserve fibers.

- There is a short exact sequence

$$1 \to \pi_1(S_g) \to \mathcal{MCG}(S_g^1) \to \mathcal{MCG}(S_g) \to 1$$

- $\mathcal{MCG}(S_g^1) \to \mathcal{MCG}(S_g)$ is the map that "forgets the puncture.

- $\pi_1(S_g) \to \mathcal{MCG}(S_g^1)$ is the "push" map, which isotopes the base point around a loop, at the end of the isotopy defining a map of S_g taking the base point to itself; then remove the base point to define a mapping class on S_g^1.

Theorem 2. *Every quasi-isometry of $\mathcal{MCG}(S_g^1)$ coarsely preserves the system of cosets of $\pi_1(S_g)$.*

Proof: Strategy: represent the short exact sequence by a fibration

$$\mathbf{H}^2 \to E \to B$$

as above, where

- $E =$ model space for $\mathcal{MCG}(S_g^1)$,

- $B =$ model space for $\mathcal{MCG}(S_g)$

- Translation of the theorem: every quasi-isometry of E coarsely preserves the fibers.

- Applying Whyte's theorem, suffices to prove:

- Top dimensional classes in B coarsely separate points.

Dimension of $\mathcal{MCG}(S_g)$. One would expect that the top dimension in which $H_n^{\mathsf{uf}}(G)$ is nontrivial would be

$$n = \mathsf{vcd}(G)$$

So we need the following formula of John Harer:

$$\mathsf{vcd}(\mathcal{MCG}(S_g)) = 4g - 5$$

Harer's proof of the upper bound: $\mathsf{vcd}(\mathcal{MCG}(S_g)) \leq 4g - 5$.

From the short exact sequence, in which

$$\mathsf{cd}(\mathsf{kernel}) = \dim(\mathbf{H}^2) = 2$$

it suffices to prove

$$\mathsf{vcd}(\mathcal{MCG}(S_g^1)) \leq 4g - 3$$

Harer constructs a contractible complex of dimension $4g-3$ which is a model space for $\mathcal{MCG}(S_g^1)$: the complex of "filling arc systems" of the once punctured surface.

Remarks:

- Harer does *not* directly construct a $4g-3$ dimensional model space for $\mathcal{MCG}(S_g)$.
- Thurston, in his 3 page 1986 preprint "A spine for the Teichmüller space of a closed surface", does construct a model space for $\mathcal{MCG}(S_g)$.
- With some work, I can prove Thurston's spine in genus 2 is indeed equal to $4g-5=8-3=3$.
- But I am unable to prove that Thurston's spine in genus ≥ 3 has dimension $4g-5$, and I think it may be false.
- Ultimately we will depend on the Eilenberg-Ganea-Wall theorem, which is why we need to compute the vcd.

Geoff Mess' proof of the lower bound:

$$\text{vcd}(\mathcal{MCG}(S_g)) \geq 4g - 5$$

$\exists$ subgroup $M_g < \mathcal{MCG}(S_g)$ which is PD of dimension $4g-5$, in fact, $M_g = \pi_1$ of a compact, aspherical $4g-5$ manifold: a *Mess subgroup* of $\mathcal{MCG}(S_g)$.

Construction of Mess subgroups by induction on genus.

Base case: Genus 2

- With $g = 2$, we have $4g - 5 = 3$, so we need a 3-dimensional subgroup of $\mathcal{MCG}(S_2)$.

- Take a curve family $\{c_1, c_2, c_3\} \subset S_2$ consisting of three pairwise disjoint, pairwise nonisotopic curves. The Dehn twists about c_1, c_2, c_3 generate a rank 3 free abelian group.

• Up to the action of $\mathcal{MCG}(S_2)$, there are two orbits of such curve families, depending on whether or not some curve in the family separates. So, there are two conjugacy classes of Mess subgroups in $\mathcal{MCG}(S_2)$.

Induction step:

• Let M_{g-1} = Mess subgroup in $\mathcal{MCG}(S_{g-1})$.

• So, M_{g-1} is a PD group of dimension $4(g-1)-5$.

• Consider the SES

$$1 \to \pi_1(S_{g-1}) \to \mathcal{MCG}(S^1_{g-1}) \to \mathcal{MCG}(S_{g-1}) \to 1$$

• Let M'_{g-1} = preimage of M_{g-1}, so we get

$$1 \to \pi_1(S_{g-1}) \to M'_{g-1} \to M_{g-1} \to 1$$

• So, M'_{g-1} is PD of dimension $4(g-1)-5+2$.

• Let $S_{g,1}$ = surface S_g with a hole removed, and with one boundary component.

• There is a central extension

$$1 \to \mathbf{Z} \to \mathcal{MCG}(S_{g,1}) \to \mathcal{MCG}(S_g^1) \to 1$$

obtained by collapsing the hole to a punctuure.

• Let M''_{g-1} = preimage of M'_{g-1}, and we get

$$1 \to \mathbf{Z} \to M''_{g-1} \to M'_{g-1} \to 1$$

• So, M''_{g-1} is PD of dimension $4(g-1)-5+3$.

• Attach a handle (a one-holed torus) to $S_{g,1}$ to get S_{g+1}, so we get an embedding

$$\mathcal{MCG}(S_{g,1}) \to \mathcal{MCG}(S_{g+1})$$

• Pick a simple closed curve c in the handle, so the Dehn twist τ_c commutes with $\mathcal{MCG}(S_{g,1})$.

• Let $M_g = M''_{g-1} \times \langle \tau_c \rangle$.

• So, M_g is PD of dimension $4(g-1)-5+4 = 4g-5$.

This finishes Mess' proof that vcd$(\mathcal{MCG}(S_g)) \geq 4g-5$.

Remarks:

• The construction of M_g is completely determined by the isotopy type of a certain filtration of S_g by subsurfaces.

• There are only finitely many such isotopy types up to the action of $\mathcal{MCG}$, and so there are only finitely many conjugacy classes of Mess subgroups.

• Let Stab(c) be the stabilizer group of the closed curve c picked in the last step, and so we have

$$M_g \subset \text{Stab}(c)$$

Model spaces

• Trick: for the moment, we won't actually work with a model space for $\mathcal{MCG}(S_g)$, instead we'll work with a model space for a finite index, torsion free subgroup $\Gamma_g < \mathcal{MCG}(S_g)$. This is OK because the inclusion $\Gamma_g \hookrightarrow \mathcal{MCG}(S_g)$ is a quasi-isometry.

• Reason for doing this: we need a contractible model space of the correct dimension $4g-5$. Don't know this exists for $\mathcal{MCG}(S_g)$, but it does exist for Γ_g, by standard results.

• Pick a torsion free, finite index subgroup $\Gamma_g < \mathcal{MCG}(S_g)$ and so

$$\mathrm{cd}(\Gamma) = 4g - 5$$

• By the Eilenberg-Ganea-Wall theorem, there exists a model space E for Γ_g of dimension $4g-5$.

• Given a Mess subgroup $M < \mathcal{MCG}(S_g)$, the intersection

$$M' = M \cap \Gamma_g$$

has finite index in M.

• So, M' is still PD of dimension $4g-5$.

• The complex E/M' is a $K(M',1)$ space of dimension $4g-5$.

• So, the (ordinary) fundamental class of $H_{4g-5}(M')$ is represented by a unique $4g-5$ cycle in E/M'.

• This cycle lifts to a $4g-5$ dimensional, uniformly finite cycle in E; call this a *Mess cycle* in E.

• It suffices to prove that the collection of Mess cycles coarsely separates points in E.

Passage to left cosets

- We now pass from Mess cycles to left cosets of Mess subgroups, as follows.

- Although $\mathcal{MCG}$ does not act on E, it does quasi-act, which is good enough.

- The quasi-action of $\mathcal{MCG}$ permutes the Mess cycles.

- There is a bijection between Mess subgroups and Mess cycles: each Mess subgroup M corresponds to a unique Mess cycle c such that M that (coarsely) stabilizes c.

- If M (coarsely) stabilizes c and if $\phi \in \mathcal{MCG}(S_g)$ then $\phi M \phi^{-1}$ (coarsely) stabilizes $\phi(c)$.

- Pick representatives $M_1, \ldots, M_k$ of the finitely many conjugacy classes of Mess subgroups.

- Follows that, under the quasi-isometry $E \to \mathcal{MCG}(S_g)$, Mess cycles correspond to left cosets in $\mathcal{MCG}(S_g)$ of $M_1, \ldots, M_k$.

- So, it suffices to show that left cosets of $M_1, \ldots, M_k$ coarsely separate points in $\mathcal{MCG}(S_g)$.

Passage to curve stabilizers

- Each Mess subgroup M_i fixes some curve c_i, and so $M_i < \text{Stab}(c_i)$.

- Thus, each left coset of M_i is contained in a left coset of $\text{Stab}(c_i)$.

- So, choosing curves $c_0, \ldots, c_n$ representing the orbits of simple closed curves, it suffices to prove that the left cosets of the groups $\text{Stab}(c_i)$ coarsely separate points in $\mathcal{MCG}$.

New model space

- We now switch to a new model space Γ, no longer contractible. We will pass from left cosets of the groups Stab(c_i) to subsets of the new model space Γ.

- Model space: a graph Γ. Vertices are pairs (C, D) where each of C, D is a pairwise disjoint curve system, the systems C, D jointly fill the surface, and each component of $S - (C \cup D)$ is a hexagon. This implies that $\mathcal{MCG}$ acts on the vertex set with finitely many orbits.

- Since $\mathcal{MCG}$ is finitely generated, and since there are finitely many orbits of vertices, it follows that we can attach edges in an $\mathcal{MCG}$-equivariant way so that the graph Γ is connected and has finitely many orbits of edges. There's probably some nice scheme for attaching edges, based on low intersection numbers, but it's not necessary. The graph Γ is now quasi-isometric to $\mathcal{MCG}$.

- Given a curve c, define Γ_c to be the subgraph of Γ spanned by vertices (C, D) such that $c \in C \cup D$.

- Passing from left cosets of curve stabilizers to the sets Γ_c, our ultimate goal is to show that the system of subgraphs Γ_c, one for each curve c, coarsely separates points in Γ.

- Given vertices (C, D) and (C', D') which are very far from each other, I'll pick a curve c in $C \cup D$ and show that (C', D') is far from Γ_c. This is enough, because (C, D) is actually in Γ_c.

- Since (C,D) and (C',D') are very far from each other, there exists $c \in C \cup D$ and $c' \in C' \cup D'$ such that the intersection number $< c,c' >$ is very large (Proof: fixing (C,D), if all such intersection numbers $< c,c' >$ are uniformly small, then there is a uniform cardinality to the number of possible (C',D'), so the distance from (C,D) to (C',D') is uniformly bounded).

- Consider now any curve system (C_1,D_1) in Γ_c, meaning that (C_1,D_1) contains c. The curve $c \in C_1 \cup D_1$ has very large intersection number with the curve $c' \in C' \cup D'$. It follows that (C_1,D_1) and (C',D') are far from each other (Proof: if (C_1,D_1) and (C',D') are close, there is a uniform bound to the intersection number of a curve in $C_1 \cup D_1$ with a curve in (C',D').

This completes the proof that quasi-isometries of $\mathcal{MCG}(S_g^1)$ coarsely preserve fibers.