Homology and dynamics in quasi-isometric rigidity
Lecture Notes, Part III
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Last time: Proving that quasi-isometries coarsely preserve fibers

Given:
- fiber bundle $F \to E \to B$, with fiber $F_x$ over each $x \in B$.
  - $d = \dim(B)$, $n = \dim(F)$, $d + n = \dim(E)$.
- $E, B$ are UC, ULF simplicial complexes
- $\pi$ is a simplicial map.
- Each fiber $F_x$ is a manifold, of dimension $n$.
- For each vertex $x$, the subcomplex $F_x$ is UC, with gauge independent of $x$.
- Top dimensional, uniformly finite homology classes in $B$ coarsely separate points.
Here is a definition of the last property (the definition I gave in the lecture for Part II was a tad too strong, and is now replaced by a more appropriate, weaker form of the definition):

**Top-d classes in** \( H_d^{uf}(B) \) **coarsely separate points:**

\[ \exists r > 0 \text{ so that } \forall s > 0 \exists D > 0 \text{ so that:} \]

for any \( x, y \in B \) with \( d(x, y) > D \), there is a top dimensional class \( c \in H_d^{uf}(B) \) such that

\[
d(\text{supp}(c), x) \leq r \quad \text{and} \quad d(\text{supp}(c), y) > s
\]

With this definition, what was proved last time was:

**Theorem 1 (Whyte).** If top dimensional classes in \( B \) coarsely separate points, then quasi-isometries of \( E \) coarsely preserve fibers.
Mapping class groups

Today: I'll explain Whyte's method for using Mess subgroups of mapping class groups to verify that quasi-isometries of $\text{MCG}(S^1_g)$ do, indeed, coarsely preserve fibers.

- There is a short exact sequence
  \[ 1 \to \pi_1(S_g) \to \text{MCG}(S^1_g) \to \text{MCG}(S_g) \to 1 \]

- $\text{MCG}(S^1_g) \to \text{MCG}(S_g)$ is the map that “forgets the puncture.

- $\pi_1(S_g) \to \text{MCG}(S^1_g)$ is the “push” map, which isotopes the base point around a loop, at the end of the isotopy defining a map of $S_g$ taking the base point to itself; then remove the base point to define a mapping class on $S^1_g$.

**Theorem 2.** Every quasi-isometry of $\text{MCG}(S^1_g)$ coarsely preserves the system of cosets of $\pi_1(S_g)$. 
Proof: Strategy: represent the short exact sequence by a fibration

\[ \mathbb{H}^2 \to E \to B \]

as above, where

- \( E = \) model space for \( \mathcal{MCG}(S^1_g) \),

- \( B = \) model space for \( \mathcal{MCG}(S_g) \)

Translation of the theorem: every quasi-isometry of \( E \) coarsely preserves the fibers.

- Applying Whyte’s theorem, suffices to prove:

  - Top dimensional classes in \( B \) coarsely separate points.
Dimension of $\mathcal{MCG}(S_g)$. One would expect that the top dimension in which $H_n^{uf}(G')$ is nontrivial would be

$$n = \text{vcd}(G')$$

So we need the following formula of John Harer:

$$\text{vcd}(\mathcal{MCG}(S_g)) = 4g - 5$$

Harer’s proof of the upper bound: $\text{vcd}(\mathcal{MCG}(S_g)) \leq 4g - 5$.

From the short exact sequence, in which

$$\text{cd(kernel)} = \dim(H^2) = 2$$

it suffices to prove

$$\text{vcd}(\mathcal{MCG}(S^1_g)) \leq 4g - 3$$

Harer constructs a contractible complex of dimension $4g - 3$ which is a model space for $\mathcal{MCG}(S^1_g)$: the complex of “filling arc systems” of the once punctured surface.
Remarks:

- Harer does not directly construct a $4g - 3$ dimensional model space for $\mathcal{MCG}(S_g)$.
- Thurston, in his 3 page 1986 preprint “A spine for the Teichmüller space of a closed surface”, does construct a model space for $\mathcal{MCG}(S_g)$.
- With some work, I can prove Thurston’s spine in genus 2 is indeed equal to $4g - 5 = 8 - 3 = 3$.
- But I am unable to prove that Thurston’s spine in genus $\geq 3$ has dimension $4g - 5$, and I think it may be false.
- Ultimately we will depend on the Eilenberg-Ganea-Wall theorem, which is why we need to compute the vcd.
Geoff Mess’ proof of the lower bound:

\[ \text{vcd}(\mathcal{MCG}(S_g)) \geq 4g - 5 \]

\exists \text{ subgroup } M_g < \mathcal{MCG}(S_g) \text{ which is PD of dimension } 4g - 5, \text{ in fact, } M_g = \pi_1 \text{ of a compact, aspherical } 4g - 5 \text{ manifold: a Mess subgroup of } \mathcal{MCG}(S_g).

Construction of Mess subgroups by induction on genus.

**Base case: Genus 2**

- With \( g = 2 \), we have \( 4g - 5 = 3 \), so we need a 3-dimensional subgroup of \( \mathcal{MCG}(S_2) \).

- Take a curve family \( \{c_1, c_2, c_3\} \subset S_2 \) consisting of three pairwise disjoint, pairwise nonisotopic curves. The Dehn twists about \( c_1, c_2, c_3 \) generate a rank 3 free abelian group.
• Up to the action of $\mathcal{MCG}(S_2)$, there are two orbits of such curve families, depending on whether or not some curve in the family separates. So, there are two conjugacy classes of Mess subgroups in $\mathcal{MCG}(S_2)$.

**Induction step:**

• Let $M_{g-1} = \text{Mess subgroup in } \mathcal{MCG}(S_{g-1})$.

• So, $M_{g-1}$ is a PD group of dimension $4(g - 1) - 5$.

• Consider the SES

$$1 \rightarrow \pi_1(S_{g-1}) \rightarrow \mathcal{MCG}(S_{g-1}^1) \rightarrow \mathcal{MCG}(S_{g-1}) \rightarrow 1$$

• Let $M'_{g-1} = \text{preimage of } M_{g-1}$, so we get

$$1 \rightarrow \pi_1(S_{g-1}) \rightarrow M'_{g-1} \rightarrow M_{g-1} \rightarrow 1$$
• So, $M'_{g-1}$ is PD of dimension $4(g - 1) - 5 + 2$.

• Let $S_{g,1} = \text{surface } S_g \text{ with a hole removed, and with one boundary component.}$

• There is a central extension

$$1 \to \mathbb{Z} \to \text{MCG}(S_{g,1}) \to \text{MCG}(S^1_g) \to 1$$

obtained by collapsing the hole to a puncture.

• Let $M''_{g-1} = \text{preimage of } M'_{g-1}$, and we get

$$1 \to \mathbb{Z} \to M''_{g-1} \to M'_{g-1} \to 1$$

• So, $M''_{g-1}$ is PD of dimension $4(g - 1) - 5 + 3$.

• Attach a handle (a one-holed torus) to $S_{g,1}$ to get $S_{g+1}$, so we get an embedding

$$\text{MCG}(S_{g,1}) \to \text{MCG}(S_{g+1})$$
• Pick a simple closed curve $c$ in the handle, so the Dehn twist $\tau_c$ commutes with $\mathcal{MCG}(S_g,1)$.

• Let $M_g = M''_{g-1} \times \langle \tau_c \rangle$.

• So, $M_g$ is PD of dimension $4(g-1) - 5 + 4 = 4g - 5$.

This finishes Mess’ proof that $\text{vcd}(\mathcal{MCG}(S_g)) \geq 4g - 5$.

Remarks:

• The construction of $M_g$ is completely determined by the isotopy type of a certain filtration of $S_g$ by sub-surfaces.

• There are only finitely many such isotopy types up to the action of $\mathcal{MCG}$, and so there are only finitely many conjugacy classes of Mess subgroups.

• Let $\text{Stab}(c)$ be the stabilizer group of the closed curve $c$ picked in the last step, and so we have

$$M_g \subset \text{Stab}(c)$$
Model spaces

• Trick: for the moment, we won’t actually work with a model space for $\mathcal{MCG}(S_g)$, instead we’ll work with a model space for a finite index, torsion free subgroup $\Gamma_g < \mathcal{MCG}(S_g)$. This is OK because the inclusion $\Gamma_g \hookrightarrow \mathcal{MCG}(S_g)$ is a quasi-isometry.

• Reason for doing this: we need a contractible model space of the correct dimension $4g - 5$. Don’t know this exists for $\mathcal{MCG}(S_g)$, but it does exist for $\Gamma_g$, by standard results.

• Pick a torsion free, finite index subgroup $\Gamma_g < \mathcal{MCG}(S_g)$, and so

$$\text{cd}(\Gamma) = 4g - 5$$

• By the Eilenberg-Ganea-Wall theorem, there exists a model space $E$ for $\Gamma_g$ of dimension $4g - 5$. 
Given a Mess subgroup $M < \mathcal{MCG}(S_g)$, the intersection

$$M' = M \cap \Gamma_g$$

has finite index in $M$.

So, $M'$ is still PD of dimension $4g - 5$.

The complex $E/M'$ is a $K(M', 1)$ space of dimension $4g - 5$.

So, the (ordinary) fundamental class of $H_{4g-5}(M')$ is represented by a unique $4g - 5$ cycle in $E/M'$.

This cycle lifts to a $4g - 5$ dimensional, uniformly finite cycle in $E$; call this a Mess cycle in $E$.

It suffices to prove that the collection of Mess cycles coarsely separates points in $E$. 


Passage to left cosets

- We now pass from Mess cycles to left cosets of Mess subgroups, as follows.

- Although $\mathcal{MCG}$ does not act on $E$, it does quasi-act, which is good enough.

- The quasi-action of $\mathcal{MCG}$ permutes the Mess cycles.

- There is a bijection between Mess subgroups and Mess cycles: each Mess subgroup $M$ corresponds to a unique Mess cycle $c$ such that $M$ (coarsely) stabilizes $c$.

- If $M$ (coarsely) stabilizes $c$ and if $\phi \in \mathcal{MCG}(S_g)$ then $\phi M \phi^{-1}$ (coarsely) stabilizes $\phi(c)$. 
• Pick representatives $M_1, \ldots, M_k$ of the finitely many conjugacy classes of Mess subgroups.

• Follows that, under the quasi-isometry $E \to \mathcal{MCG}(S_g)$, Mess cycles correspond to left cosets in $\mathcal{MCG}(S_g)$ of $M_1, \ldots, M_k$.

• So, it suffices to show that left cosets of $M_1, \ldots, M_k$ coarsely separate points in $\mathcal{MCG}(S_g)$. 
Passage to curve stabilizers

- Each Mess subgroup \( M_i \) fixes some curve \( c_i \), and so \( M_i < \text{Stab}(c_i) \).

- Thus, each left coset of \( M_i \) is contained in a left coset of \( \text{Stab}(c_i) \).

- So, choosing curves \( c_0, \ldots, c_n \) representing the orbits of simple closed curves, it suffices to prove that the left cosets of the groups \( \text{Stab}(c_i) \) coarsely separate points in \( \text{MCG} \).
New model space

- We now switch to a new model space $\Gamma$, no longer contractible. We will pass from left cosets of the groups $\text{Stab}(c_i)$ to subsets of the new model space $\Gamma$.

- Model space: a graph $\Gamma$. Vertices are pairs $(C, D)$ where each of $C, D$ is a pairwise disjoint curve system, the systems $C, D$ jointly fill the surface, and each component of $S - (C \cup D)$ is a hexagon. This implies that $\text{MCG}$ acts on the vertex set with finitely many orbits.

- Since $\text{MCG}$ is finitely generated, and since there are finitely many orbits of vertices, it follows that we can attach edges in an $\text{MCG}$-equivariant way so that the graph $\Gamma$ is connected and has finitely many orbits of edges. There’s probably some nice scheme for attaching edges, based on low intersection numbers,
but it’s not necessary. The graph $\Gamma$ is now quasi-isometric to $\mathcal{MCG}$.

- Given a curve $c$, define $\Gamma_c$ to be the subgraph of $\Gamma$ spanned by vertices $(C, D)$ such that $c \in C \cup D$.

- Passing from left cosets of curve stabilizers to the sets $\Gamma_c$, our ultimate goal is to show that the system of subgraphs $\Gamma_c$, one for each curve $c$, coarsely separates points in $\Gamma$.

- Given vertices $(C, D)$ and $(C', D')$ which are very far from each other, I’ll pick a curve $c$ in $C \cup D$ and show that $(C', D')$ is far from $\Gamma_c$. This is enough, because $(C, D)$ is actually in $\Gamma_c$. 
Since \((C, D)\) and \((C', D')\) are very far from each other, there exists \(c \in C \cup D\) and \(c' \in C' \cup D'\) such that the intersection number \(<c, c'>\) is very large (Proof: fixing \((C, D)\), if all such intersection numbers \(<c, c'>\) are uniformly small, then there is a uniform cardinality to the number of possible \((C', D')\), so the distance from \((C, D)\) to \((C', D')\) is uniformly bounded).

Consider now any curve system \((C_1, D_1)\) in \(\Gamma_c\), meaning that \((C_1, D_1)\) contains \(c\). The curve \(c \in C_1 \cup D_1\) has very large intersection number with the curve \(c' \in C' \cup D'\). It follows that \((C_1, D_1)\) and \((C', D')\) are far from each other (Proof: if \((C_1, D_1)\) and \((C', D')\) are close, there is a uniform bound to the intersection number of a curve in \(C_1 \cup D_1\) with a curve in \((C', D')\).

This completes the proof that quasi-isometries of \(\mathcal{MCG}(S^1_g)\) coarsely preserve fibers.