Homology and dynamics in quasi-isometric rigidity Lecture Notes, Part IV Durham, July 8, 2003

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Last time:

Theorem 1. *For $g \geq 2$, every quasi-isometry of $\mathcal{MCG}(S_g^1)$ coarsely preserves the cosets of $\pi_1(S_g)$ in the short exact sequence*

$$1 \to \pi_1(S_g) \hookrightarrow \mathcal{MCG}(S_g^1) \xrightarrow{\pi} \mathcal{MCG}(S_g) \to 1$$

Today, use the conclusion of the above to prove:

Theorem 2. *Every quasi-isometry of $\mathcal{MCG}(S_g^1)$ is close to left translation by some element of the group. In particular, the injection $\mathcal{MCG}(S_g^1) \to \mathrm{QI}(\mathcal{MCG}(S_g^1))$ is an isomorphism.*

Precisely:

- $\forall K \geq 1$, $C \geq 0$ $\exists A \geq 0$ such that if ϕ is a K, C quasi-

isometry of $\mathcal{MCG}(S_g^1)$ then there exists $\gamma \in \mathcal{MCG}(S_g^1)$ such that

$$d_{\mathsf{sup}}(\phi, L_\gamma) < A$$

Then, as explained earlier, we have

Corollary 3. *If G is a f.g. group quasi-isometric to $\mathcal{MCG}(S_g^1)$ then there exists a homomorphism $G \to \mathcal{MCG}(S_g^1)$ with finite kernel and finite index image.*

The methods of proof are very similar to the following:

Theorem 4 (Farb-M). *Let F be a Schottky subgroup of $\mathcal{MCG}(S_g)$ and consider the split extension*

$$1 \to \pi_1(S_g) \to \Gamma_F \to F \to 1$$

The injection $\Gamma_F \to \mathrm{QI}(\Gamma_F)$ has finite index image; more precisely . . .

In that proof:

- Using a tree as a model space for F, proved that every quasi-isometry of Γ_F coarsely preserves fibers.
- Then used pseudo-Anosov dynamics, as we will today.

Background of the proof: Teichmüller space and its canonical $\mathbf{H}^2$ bundle.

$$\begin{aligned}
\mathcal{T}_g &= \text{Teichmüller space of } S_g \\
&= \{\text{hyperbolic structures on } S_g\}/\text{isotopy} \\
&= \{\text{conformal structures on } S_g\}/\text{isotopy} \\
\mathcal{T}_g^1 &= \text{Teichmüller space of } S_g^1 \\
&= \{(\sigma, p) \,\big|\, \sigma \in \mathcal{T}_g, \quad p \in \widetilde{\sigma} \approx \mathbf{H}^2\}
\end{aligned}$$

There is a fiber bundle structure

$$\mathbf{H}^2 \to \mathcal{T}_g^1 \to \mathcal{T}_g$$

on which the short exact sequence

$$1 \to \pi_1(S_g) \to \mathcal{MCG}(S_g^1) \to \mathcal{MCG}(S_g) \to 1$$

acts.

The actions of $\mathcal{MCG}(S_g^1)$ on $\mathcal{T}_g^1$ and of $\mathcal{MCG}(S_g)$ on $\mathcal{T}_g$ are *not* cocompact.

But there is a cocompact equivariant spine

$$\mathcal{S}_g \subset \mathcal{T}_g$$

whose inverse image is a cocompact spine

$$\mathcal{S}_g^1 \subset \mathcal{T}_g^1$$

and we have a fibration

$$\mathbf{H}^2 \to \mathcal{S}_g^1 \to \mathcal{S}_g$$

on which the short exact sequence acts.

By cocompactness, the $\mathcal{S}$'s are model spaces for the $\mathcal{MCG}$'s.

$\mathbf{H}^2$ **bundles over lines:** We know a QI $\phi \colon S_g^1 \to S_g^1$ of coarsely respects fibers, so it coarsely respects saturated sets, so it induces a quasi-isometry of $\mathcal{S}_g$, also denoted ϕ.

Consider bi-infinite, proper paths $\ell \colon \mathbf{R} \to \mathcal{T}_g$, with image often in $\mathcal{S}_g$. ℓ is always piecewise smooth and Lipschitz. Let

$$\Sigma_\ell = \pi^{-1}(\ell) \subset \mathcal{T}_g^1$$

so we have an $\mathbf{H}^2$ bundle over the line ℓ:

$$\mathbf{H}^2 \to \Sigma_\ell \to \ell$$

Σ_ℓ has a natural metric: $\mathbf{H}^2$ metric on fibers, plus metric on $\mathbf{R}$ factor. (There are some choices, but the metric is natural up to QI).

Example: if ℓ is piecewise geodesic then the metric on Σ_ℓ is just the pullback of the metric on $\mathcal{T}_g^1$.

If $\ell, \ell' \colon \mathbf{R} \to \mathcal{S}_g$ are "fellow travellers", then the induced map $\Sigma_\ell \to \Sigma_{\ell'}$ is a quasi-isometry.

A quasi-isometry $\phi \colon \mathcal{S}_g^1 \to \mathcal{S}_g^1$ induces a coarsely fiber preserving quasi-isometry

$$\Sigma_\ell \to \Sigma_{\phi(\ell)}$$

Example: Suppose that ℓ is the axis of a pseudo-Anosov diffeomorphism, ore more generally, that ℓ fellow travels such an axis. Thurston's hyperbolization theorem for fibered 3-manifolds implies:

$$\Sigma_\ell \overset{\mathrm{QI}}{\approx} \mathbf{H}^3$$

and so Σ_ℓ is a Gromov hyperbolic metric space.

Background: Teichmüller geodesics and pseudo-Anosov dynamics

A *quadratic differential* on S_g is a transverse pair of measured foliations:

$$q = (\mathcal{F}_u, \mathcal{F}_s)$$

Each quadratic differential q determines a singular Euclidean metric, which determines conformal structure with removable singularities, which determines a point $\sigma(q) \in \mathcal{T}_g$.

For each $t \in \mathbf{R}$ define

$$q_t = (e^{-t}\mathcal{F}_u, e_t\mathcal{F}_s)$$

The path

$$\gamma_q = \{t \mapsto \sigma(q_t) \,\Big|\, t \in \mathbf{R}\}$$

in $\mathcal{T}_g$ is the *Teichmüller geodesic* corresponding to q.

Let Σ_q^{solv} denote the hyperbolic plane bundle Σ_{γ_q} with the *singular* solv *metric*, defined by

$$e^{-2t}\, d\mathcal{F}_u^2 + e^{2t}\, d\mathcal{F}_s^2 + dt^2$$

Fact: if γ_q is cobounded then the identity map $\Sigma_{\gamma_q} \to \Sigma_q^{\text{solv}}$ is a quasi-isometry between the "natural" metric and the singular solv metric.

Pseudo-Anosov homeomorphisms, their axes, and their hidden symmetries. Here is the precise definition of pseudo-Anosov homeomorphisms and their axes.

A homeomorphism $f\colon S_g \to S_g$ is *pseudo-Anosov* if there exists a $\mathcal{T}$-geodesic $q_f = (\mathcal{F}_u, \mathcal{F}_s)$ such that f acts as a translation on γ_q. Let Σ_f denote Σ_{q_f}.

Equivalently: up to isotopy, there exists a qd $q_f = (\mathcal{F}_u, \mathcal{F}_s)$ and $\lambda > 1$ such that

$$f(\mathcal{F}_u, \mathcal{F}_s) = (\lambda^{-1}\mathcal{F}_u, \lambda\mathcal{F}_s)$$

The group $J_f = \pi_1(S_g) \rtimes_f \mathbf{Z}$ acts by isometries on Σ_f^{solv}, that is,

$$J_f < I_f = \text{Isom}(\Sigma_f^{\text{solv}}))$$

It is possible that J_f is properly contained in I_f. We can think of the elements of $I_f - J_f$ as "hidden symmetries" of f.

Explicit computation of I_f:

- There exists a maximal index orbifold subcover $S_g \to O_f$ such that f descends to a pseudo-Anosov homeomorphism of the orbifold O_f.
- Let N_f be the normalizer of f in $\mathcal{MCG}(\mathcal{O}_f)$.

Fact 5. *There is a natural extension*

$$1 \to \pi_1(\mathcal{O}_f) \to I_f \to N_f$$

In particular, I_f contains J_f with finite index.

A direct construction can be used to show:

Fact 6. *There exist pseudo-Anosov homeomorphisms with no hidden symmetries, that is, so that* $I_f = J_f$.

Strategy of the proof:

Apply various quasi-isometry invariants to the metric spaces Σ_ℓ.

Definition: the line $\ell\colon \mathbf{R} \to \mathcal{S}_g$ and its $\mathbf{H}^2$ bundle Σ_ℓ are *hyperbolic* if Σ_ℓ is a δ-hyperbolic metric space for some $\delta \geq 0$.

Since hyperbolicity is a QI-invariant, we obtain:

Fact 7 (Hyperbolic spaces are preserved). *Every quasi-isometry of* $\mathcal{MCG}(S_g^1)$ *coarsely respects the hyperbolic spaces* Σ_ℓ.

Definition: The line $\ell \subset \mathcal{T}_g$, or its $\mathbf{H}^2$ bundle Σ_ℓ, is *periodic* if ℓ is a Teichmüller geodesic and there exists $f \in \mathcal{MCG}(S_g)$ such that f preserves ℓ, acting by a translation.

Equivalently, by Thurston's hyperbolization theorem, Σ_ℓ is the universal cover of a fibered hyperbolic 3-manifold.

We say ℓ is *coarsely periodic* if it fellow travels something that is periodic.

Surprise: (coarsely) periodic hyperbolic spaces are coarsely preserved:

Theorem 8. *Every quasi-isometry of $\mathcal{MCG}(S_g^1)$ coarsely respects the periodic hyperbolic 3-manifolds Σ_ℓ.*

To be precise, given:

- *Quasi-isometry $\phi\colon \mathcal{S}_g^1 \to \mathcal{S}_g^1$*
- *Periodic line $\ell \subset \mathcal{T}_g$*
- *Line $\ell' \subset \mathcal{S}_g$ that fellow travels ℓ*

There exists a periodic line $\phi_*(\ell) \subset \mathcal{T}_g$ such that $\phi(\Sigma_{\ell'})$ has finite Hausdorff distance form $\Sigma_{\phi_*(\ell)}$.

Moreover, ϕ induces a quasi-isometry

$$\Sigma_\ell^{\mathsf{solv}} \to \Sigma_{\phi_*(\ell)}^{\mathsf{solv}}$$

which is a bounded distance from an isometry of singular solv *metrics.*

This is the heart of the matter. Before giving the proof, first let's apply it:

Proof of QI-rigidity of $\mathcal{MCG}(S_g^1)$.

Consider a QI $\phi\colon \mathcal{S}_g^1 \to \mathcal{S}_g^1$.

• Consider pseudo-Anosov homeomorphisms $f\colon S_g \to S_g$ without hidden symmetries: the group

$$J_f = \pi_1(S_g) \rtimes_f \mathbf{Z}$$

is the entire isometry group of the singular solv manifold Σ_f^{solv}.

• If f has no hidden symmetries, then each conjugate gfg^{-1} has no hidden symmetries.

• For each f without hidden symmetries, there is an f' such that ϕ takes Σ_f to $\Sigma_{f'}$. Moreover, f' has no hidden symmetries (because ϕ conjugates I_f to $I_{f'}$).

• Follows that f, f' are conjugate elements of $\mathcal{MCG}(S_g)$,

$$f' = k_f\, f\, k_f^{-1}, \qquad \text{for some} \quad k_f \in \mathcal{MCG}(S_g)$$

Proof: since ϕ conjugates I_f to $I_{f'}$, we obtain an induced mapping class from $S_g = \mathbf{H}^2/\pi_1(S_g) \to \mathbf{H}^2/\pi_1(S_g) = S_g$.

• Moreover, the induced quasi-isometry $\Sigma_f \to \Sigma_{f'}$ isometry itself is within bounded distance from the left action of some $\theta_f \in \mathcal{MCG}(S_g^1)$, which is a lift of $k_f \in \mathcal{MCG}(S_g)$.

Note: θ_f can be regarded as a self-QI on $\mathbf{H}^2$, and is determined by its boundary values.

• Take f_1, f_2 to be two pseudo-Anosov homeomorphisms, without hidden symmetries, and suppose that their axes A_1, A_2 which are close to some point x. The induced quasi-isometry $F_x \to F_{\phi(x)}$ must agree with both θ_{f_1} and θ_{f_2}.

• This is only possible if $\theta_{f_1} = \theta_{f_2}$. Proof: they have the same boundary values.

• But the union of axes for the conjugates of f come uniformly close to every point in $\mathcal{S}_g$.

• So, for $x, x' \in \mathcal{S}_g$, take a chain $f_1, f_2, \ldots, f_n$ of conjugates of f so that the axes of these conjugates go, by bounded steps, from a bounded neighborhood of x to a bounded neighborhood of x'. Follows that both of the induced maps $F_x \to F_{\phi(x)}$ and $F_{x'} \to F_{\phi(x')}$ agree with $\theta_{f_1} = \theta_{f_2} = \cdots = \theta_{f_n}$.

• Thus, $\theta = \theta_k \in \mathcal{MCG}(S_g^1)$ is independent of k and ϕ is close to left translation by θ.

$\diamondsuit$

Proof that Periodic hyperbolic spaces are coarsely preserved

Step 1

Theorem 9 (M; Bowditch). *A line $\ell \colon \mathbf{R} \to \mathcal{T}_g$ is hyperbolic if and only if there exists a cobounded Teichmüller geodesic $\gamma \colon \mathbf{R} \to \mathcal{T}_g$ such that, up to quasi-isometric reparameterization, ℓ fellow travels γ.*

For example, a pseudo-Anosov axis is a cobounded Teichmüller geodesic, forming a countable family. However, there are uncountably many cobounded Teichmüller geodesics.

One minute proof. (every theorem should have a one minute proof, a five minute proof, . . .)

• "Hyperbolicity" means "exponential divergence of geodesics".

• Ordinarily this applies to geodesic rays passing transversely through spheres, but it also applies to geodesics in Σ_ℓ passing transversely through fibers $F_t = \ell^{-1}(t)$.

• Certain geodesics in the fibers $F_t = \ell^{-1}(t)$ are stretched exponentially as $t \to \infty$, forming the unstable foliation $\mathcal{F}_u$, and certain other geodesics in F_t are stretched exponentially as $t \to -\infty$, forming the stable foliation $\mathcal{F}_s$.

• Taking $q = (\mathcal{F}_u, \mathcal{F}_s)$, a compactness argument shows that ℓ fellow travels the Teichmüller geodesic σ_q. $\diamondsuit$

Step 2 Given a cobounded Teichmüller geodesic γ, define $\mathrm{QI}_f(\Sigma_\gamma^{\mathsf{solv}})$ to be the group of "fiber respecting quasi-isometries" of $\Sigma_\gamma^{\mathsf{solv}}$.

- The group $\mathrm{QI}_f(\Sigma_\gamma^{\mathsf{solv}})$ is an invariant of $\Sigma_\gamma^{\mathsf{solv}}$ up to fiber respecting quasi-isometry: a fiber respecting quasi-isometry $\Sigma_\gamma^{\mathsf{solv}} \to \Sigma_{\gamma'}^{\mathsf{solv}}$ induces an isomorphism of QI_f.

- The group $\mathrm{QI}_f(\Sigma_\gamma^{\mathsf{solv}})$ contains $\pi_1(S_g)$.

Step 3 Reduction to key fact:

$$\mathrm{QI}_f(\Sigma_\gamma) = \mathrm{Isom}(\Sigma_\gamma^{\mathsf{solv}})$$

Noting that γ is periodic if and only if $\mathrm{Isom}(\Sigma_\gamma^{\mathsf{solv}})$ contains $\pi_1(S_g)$ with infinite index, the latter clearly depends only on QI_f, and so periodicity of γ is a fiber respecting quasi-isometry invariant of γ.

Proof of key fact: Farb and I gave a proof that uses Thurston's hyperbolization theorem for fibered 3-manifolds together with Rich Schwartz' geodesic pattern rigitidy theorem. I would like to be able to give a proof of this fact based solely on pseudo-Anosov dynamics, and I may be able to do this, but in lieu of total confidence in the details, let me sketch the original proof of Farb and myself.

The singular lines of $\Sigma_\gamma^{\mathsf{solv}}$ form a collection of singular solv geodesics intersecting the fibers at right angles; let Ω denote this collection of geodesics. If ϕ is a fiber respecting quasi-isometry of $\Sigma_\gamma^{\mathsf{solv}}$, then ϕ respects leaves of f^s and f^u as noted earlier, and in fact ϕ respects the suspensions of these leaves. It follows that ϕ respects Ω, because the singular lines in Ω are precisely the sets which, coarsely, are intersections of three or more suspensions of leaves of f^s (or of f^u) whose pairwise intersections are unbounded. This way of characterizing the lines of Ω is clearly invariant under fiber respecting quasi-isometry, and so Ω is respected.

By Thurston's hyperbolization theorem, there is an I_γ-equivariant $\mathbf{H}^3$ metric on Σ_γ. The lines in Ω can be straightened to hyperbolic geodesics, which are evidently invariant under I_γ. This is exactly the setup of Schwartz' theorem, whose conclusion is that the group of quasi-isometries of $\mathbf{H}^3$ that coarsely respects Ω contains I_γ with finite index, that is, $\mathrm{QI}_f(\Sigma_\gamma)$ contains I_γ with finite index. But then an easy argument shows that $\mathrm{QI}_f(\Sigma_\gamma)$ must actually consist of singular solv isometries.

This finishes the proof of QI-rigidity of $\mathcal{MCG}(S_g^1)$.